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q-Wakimoto Modules and Integral Formulae of Solutions of the Quantum Knizhnik–Zamolodchikov Equations

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Abstract. Matrix elements of intertwining operators between q-Wakimoto modules associated to the tensor product of representations of $U_q(\widehat{sl_2})$ with arbitrary spins are studied. It is shown that they coincide with the Tarasov–Varchenko's formulae of the solutions of the qKZ equations. The result generalizes that of the previous paper [Kuroki K., Nakayashiki A., *SIGMA* **4** (2008), 049, 13 pages].

Key words: free field; vertex operator; qKZ equation; q-Wakimoto module

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1 Introduction

In [8] the integral formulae of the quantum Knizhnik–Zamolodchikov (qKZ) equations [3] for the tensor product of spin 1/2 representation of $U_q(sl_2)$ arising from q-Wakimoto modules have been studied. The formulae are identified with those of Tarasov–Varchenko's formulae. The aim of this paper is to generalize the results to the case of tensor product of representations with arbitrary spins.

It is known that certain matrix elements of intertwining operators between q-Wakimoto modules satisfy the qKZ equation [3, 10]. Thus it is interesting to compute those matrix elements explicitly. In [5] two kinds of intertwining operators were introduced, type I and type II. They were defined according as the position of evaluation representations. In the application to the study of solvable lattice models two types of operators have their own roles. Type I and type II operators correspond to states and particles respectively. The properties of traces exhibit very different structure. However as far as the matrix elements are concerned they are not expected to be very different [5].

In [8] a computation of matrix elements has been carried out in the case of type I operator and the tensor product of 2-dimensional vector representation of $U_q(sl_2)$ generalizing the result of [10] (see the previous paper [8]). In this paper we compute matrix elements for the composition of the type I intertwining operators [5] associated to finite dimensional irreducible representations of $U_q(sl_2)$. We perform certain multidimensional integrals and sums explicitly. It is shown that the formulae thus obtained coincide with those of Matsuo [9], Tarasov and Varchenko [13] without the term corresponding to the deformed cycles.

To obtain actual matrix elements of intertwining operators it is necessary to specify certain contours of integration associated to screening operators. We do not consider this problem in this paper. To find integration contours describing each composition of intertwining operators is an important open problem. We also remark that the formulae for type II intertwining operators are not obtained in this paper. The computation of them looks quite different from that for type I case as opposed to the expectation. It is interesting to find the way to get a similar result for matrix elements in the case of type II operators.

The paper is organized in the following manner. The construction of the solutions of the qKZ equations due to Tarasov and Varchenko is reviewed in Section 2. In Section 3 a free field construction of intertwining operator is reviewed. The formulae for the matrix elements of some operators are calculated in Section 4. The main theorem of this paper is stated in this section. In Section 5 the proof of the main theorem is given. The evaluation representation of $U_q(\widehat{sl_2})$ is explicitly described in Appendix A. Appendix B gives the explicit form of the R-matrix in special cases. The explicit forms of the operators which appear in Section 3 are given in Appendix C. Appendix D contains the list of OPE's which is necessary to derive the integral formulae.

2 Tarasov–Varchenko's formulae

We review Tarasov–Varchenko's formula for solutions of the qKZ equations. In this paper we assume that q is a complex number such that $|q| < 1$. We mainly follow the notation of [13]. For a nonnegative integer l let $V^{(l)} = \bigoplus_{i=0}^{l} \mathbb{C}v_i^{(l)}$ be the $l+1$ dimensional irreducible $U_q(sl_2)$-module and $V_z^{(l)} = V^{(l)} \otimes \mathbb{C}[z, z^{-1}]$ the evaluation representation of $U_q(\widehat{sl_2})$ on $V^{(l)}$. The action of $U_q(\widehat{sl_2})$ on $V_z^{(l)}$ is given in Appendix A. Let l_1 and l_2 be nonnegative integers and $R_{l_1,l_2}(z) \in \mathrm{End}(V^{(l_1)} \otimes V^{(l_2)})$ the trigonometric quantum R-matrix uniquely determined by the following conditions:

$$(i) \quad PR_{l_1,l_2}(z) \text{ commutes with } U_q(\widehat{sl_2}),$$
$$(ii) \quad PR_{l_1,l_2}(z)\big(v_0^{(l_1)} \otimes v_0^{(l_2)}\big) = v_0^{(l_2)} \otimes v_0^{(l_1)},$$

where $P : V^{(l_1)} \otimes V^{(l_2)} \to V^{(l_2)} \otimes V^{(l_1)}$ is a linear map given by

$$P(v \otimes w) = w \otimes v.$$

The explicit form of the R-matrix is given in Appendix B in case $l_1 = 1$ or $l_2 = 1$. We set

$$\widehat{R}_{l_i,l_j}(z) = \rho_{l_i,l_j}(z)\widetilde{R}_{l_i,l_j}(z), \qquad \widetilde{R}_{l_i,l_j}(z) = (C_{l_i} \otimes C_{l_j})R_{l_i,l_j}(z)(C_{l_i} \otimes C_{l_j}),$$
$$\rho_{l_i,l_j}(z) = q^{\frac{l_i l_j}{2}} \frac{(q^{l_i+l_j+2}z^{-1}; q^4)_\infty (q^{-l_i-l_j+2}z^{-1}; q^4)_\infty}{(q^{-l_i+l_j+2}z^{-1}; q^4)_\infty (q^{l_i-l_j+2}z^{-1}; q^4)_\infty},$$
$$C_l v_\epsilon^{(l)} = v_{l-\epsilon}^{(l)} \qquad \big(v_\epsilon^{(l)} \in V^{(l)}\big),$$

where for a complex number a with $|a| < 1$

$$(z; a)_\infty = \prod_{i=0}^{\infty} \big(1 - a^i z\big).$$

Let k be a complex number. We set

$$p = q^{2(k+2)}.$$

We assume that p satisfies $|p| < 1$. Let T_j denote the p-shift operator of z_j,

$$T_j f(z_1, \ldots, z_n) = f(z_1, \ldots, pz_j, \ldots z_n).$$

Let $l_1, \ldots, l_n$ and N be nonnegative integers. The qKZ equation for a $V_{l_1} \otimes \cdots \otimes V_{l_n}$-valued function $\Psi(z_1, \ldots, z_n)$ is

$$T_j \Psi = \widehat{R}_{j,j-1}(pz_j/z_{j-1}) \cdots \widehat{R}_{j,1}(pz_j/z_1)\kappa^{\frac{h_j}{2}} \widehat{R}_{j,n}(z_j/z_n) \cdots \widehat{R}_{j,j+1}(z_j/z_{j+1})\Psi, \tag{1}$$

where κ is a complex parameter, $\widehat{R}_{i,j}(z)$ signifies that $\widehat{R}_{l_i,l_j}(z)$ acts on the i-th and j-th components of the tensor product and κ^{h_j} acts on j-th component as

$$\kappa^{\frac{h_j}{2}} v_m^{(l_j)} = \kappa^{\frac{l_j-2m}{2}} v_m^{(l_j)}.$$

We set

$$(z)_\infty = (z;p), \qquad \theta(z) = (z)_\infty \left(pz^{-1}\right)_\infty (p)_\infty.$$

Consider a sequence $(\nu) = (\nu_1, \dots, \nu_n)$ satisfying $0 \le \nu_i \le l_i$ for all i and $N = \sum_{i=1}^{n} \nu_i$. Let $r = \sharp\{i \,|\, \nu_i \neq 0\}$, $\{i \,|\, \nu_i \neq 0\} = \{k(1) < \cdots < k(r)\}$ and $n_i = \nu_{k(i)}$. We set

$$w_{(\nu)}(t,z) = \prod_{a<b} \frac{t_a - t_b}{q^{-2}t_a - t_b} \sum_{\substack{\Gamma_1 \sqcup \cdots \sqcup \Gamma_r = \{1,\dots,N\} \\ |\Gamma_s| = n_s (s=1,\dots,r)}} \left(\prod_{\substack{1 \le i < j \le r \\ a \in \Gamma_i, b \in \Gamma_j}} \frac{q^{-2}t_a - t_b}{t_a - t_b} \right)$$
$$\times \prod_{b \in \Gamma_s} \left(\frac{t_b}{t_b - q^{-l_{k(i)}} z_{k(i)}} \prod_{j<k(i)} \frac{q^{-l_j} t_b - z_j}{t_b - q^{-l_j} z_j} \right).$$

The elliptic hypergeometric space $\mathcal{F}_{\text{ell}}$ is the space of functions $W(t,z){=}W(t_1, \dots, t_N, z_1, \dots, z_n)$ of the form

$$W = Y(z)\Theta(t,z) \frac{1}{\prod_{j=1}^{n} \prod_{a=1}^{N} \theta(q^{l_j} t_a / z_j)} \prod_{1 \le a < b \le N} \frac{\theta(t_a/t_b)}{\theta(q^{-2} t_a / t_b)}$$

satisfying the following conditions:

(*i*) $Y(z)$ is meromorphic on $(\mathbb{C}^*)^n$ in $z_1, \dots, z_n$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$;

(*ii*) $\Theta(t,z)$ is holomorphic on $(\mathbb{C}^*)^{n+N}$ in $t_1, \dots, z_n$ and symmetric in $t_1, \dots, t_N$;

(*iii*) $T_a^t W/W = \kappa q^{-2N+4a-2} \prod_{i=1}^{n} q^{l_i}$, $T_j^z W/W = q^{-l_j N}$, where $T_a^t W = W(t_1, \dots, pt_a, \dots, t_N, z)$ and $T_j^z W = W(t, z_1, \dots, pz_j, \dots, z_n)$.

Define the phase function $\Phi(t,z)$ by

$$\Phi(t,z) = \left(\prod_{a=1}^{N} \prod_{i=1}^{n} \frac{(q^{l_i} t_a / z_i)_\infty}{(q^{-l_i} t_a / z_i)_\infty} \right) \left(\prod_{a<b} \frac{(q^{-2} t_a / t_b)_\infty}{(q^2 t_a / t_b)_\infty} \right).$$

For $W \in \mathcal{F}_{\text{ell}}$ let

$$I(w_{(\epsilon)}, W) = \int_{\widetilde{\mathbb{T}}^N} \prod_{a=1}^{N} \frac{dt_a}{t_a} \Phi(t,z) w_{(\epsilon)}(t,z) W(t,z), \tag{2}$$

where $\widetilde{\mathbb{T}}^N$ is a suitable deformation of the torus

$$\mathbb{T}^N = \{(t_1, \dots, t_N) \,|\, |t_i| = 1, 1 \le i \le N\},$$

specified as follows. The integrand has simple poles at

$$t_a / z_j = \left(p^s q^{-l_j}\right)^{\pm 1}, \qquad s \ge 0, \quad 1 \le a \le N, \quad 1 \le j \le n,$$

$$t_a/t_b = \left(p^s q^2\right)^{\pm 1}, \qquad s \geq 0, \quad 1 \leq a < b \leq N.$$

The contour of integration in t_a is a simple closed curve which rounds the origin in the counter-clockwise direction and separates the following two sets

$$\left\{p^s q^{-l_j} z_j, p^s q^2 t_b | s \geq 0, 1 \leq j \leq N, a < b\right\},$$
$$\left\{p^{-s} q^{l_j} z_j, p^{-s} q^{-2} t_b | s \geq 0, 1 \leq j \leq N, a < b\right\}.$$

Let L be a complex number and

$$\kappa = q^{-2\left(L + \sum_{i=1}^{n} \frac{l_i}{2} - N + 1\right)}.$$

Then

$$\Psi_W = \left(\prod_{i=1}^{n} z_i^{a_i}\right)\left(\prod_{i<j} \xi_{l_i,l_j}(z_i/z_j)\right) \sum_{(\epsilon)} I(w_{(-\epsilon)}, W) v_{\epsilon_1}^{(l_1)} \otimes \cdots \otimes v_{\epsilon_n}^{(l_n)} \tag{3}$$

is a solution of the qKZ equation (1) for any $W \in \mathcal{F}_{\mathrm{ell}}$ where $(-\epsilon) = (l_1 - \epsilon_1, \ldots, l_n - \epsilon_n)$ and

$$a_i = \frac{l_i}{2(k+2)}\left(L + \sum_{j=1}^{n} l_j - \frac{l_i}{2} - N + 1\right),$$
$$\xi_{l_i,l_j}(z) = \frac{\left(pq^{l_i+l_j+2} z^{-1}; q^4, p\right)_\infty \left(pq^{-l_i-l_j+2} z^{-1}; q^4, p\right)_\infty}{\left(pq^{l_i-l_j+2} z^{-1}; q^4, p\right)_\infty \left(pq^{-l_i+l_j+2} z^{-1}; q^4, p\right)_\infty},$$
$$(z; p, q) = \prod_{i=0}^{\infty}\prod_{j=0}^{\infty}(1 - p^i q^j z).$$

3 Free field realizations

We briefly review the free field construction of the representation of the $U_q(\widehat{sl_2})$ of level k [1, 10, 11] and intertwining operators [2, 6, 7]. We mainly follow the notation of [6]. We set

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Let k be a complex number and $\{a_n, b_n, c_n, \tilde{a}_0, \tilde{b}_0, \tilde{c}_0, Q_a, Q_b, Q_c \,|\, n \in \mathbb{Z}_{\geq 0}\}$ satisfy

$$[a_n, a_m] = \delta_{m+n,0}\frac{[(k+2)n][2n]}{n}, \qquad [\tilde{a}_0, Q_a] = 2(k+2),$$
$$[b_n, b_m] = \delta_{m+n,0}\frac{-[2n]^2}{n}, \qquad [\tilde{b}_0, Q_b] = -4,$$
$$[c_n, c_m] = \delta_{m+n,0}\frac{[2n]^2}{n}, \qquad [\tilde{c}_0, Q_c] = 4.$$

Other combinations of elements are supposed to commute. Set

$$N_\pm = \mathbb{C}[a_n, b_n, c_n \mid \pm n > 0].$$

Let r be a complex number and s an integer. The Fock module $F_{r,s}$ is defined to be the free N_- module of rank one generated by the vector $|r, s\rangle$ satisfying

$$N_+|r,s\rangle = 0, \qquad \tilde{a}_0|r,s\rangle = r|r,s\rangle, \qquad \tilde{b}_0|r,s\rangle = -2s|r,s\rangle, \qquad \tilde{c}_0|r,s\rangle = -2s|r,s\rangle.$$

We set

$$F_r = \oplus_{s\in\mathbb{Z}} F_{r,s}.$$

The right Fock module $F^\dagger_{r,s}$ and $F^\dagger_r$ are similarly defined using the vector $\langle r,s|$ satisfying the conditions

$$\langle r,s|N_- = 0, \qquad \langle r,s|\tilde{a}_0 = r\langle r,s|, \qquad \langle r,s|\tilde{b}_0 = -2s\langle r,s|, \qquad \langle r,s|\tilde{c}_0 = -2s\langle r,s|.$$

Notice that F_r and $F^\dagger_r$ have left and right $U_q(\widehat{sl_2})$-module structure respectively [10, 11].

Let

$$|L\rangle = |L,0\rangle \in F_{L,0}, \qquad \langle L| = \langle L,0| \in F^\dagger_{L,0}.$$

They become left and right highest weight vectors of $U_q(\widehat{sl_2})$ with the weight $L\Lambda_1 + (k-L)\Lambda_0$ respectively, where Λ_0 and Λ_1 are fundamental weights of $\widehat{sl_2}$.

We consider operators

$$\phi^{(l)}_m(z):\ F_{r,s} \to F_{r+l,s+l-m}, \qquad J^-(u):\ F_{r,s}\to F_{r,s+1}, \qquad S(t):\ F_{r,s}\to F_{r-2,s-1},$$

the explicit forms of which are given in Appendix C. We set

$$\phi^{(l)}_l(z) = \phi_l(z)$$

for simplicity. The operator $\phi^{(l)}_m(z)$ is used to construct the vertex operator for $U_q(\widehat{sl_2})$:

$$\phi^{(l)}(z):\ W_r \to W_{r+l}\otimes V^{(l)}_z, \qquad \phi^{(l)}(z) = \sum_{m=0}^{l} \phi^{(l)}_m(z)\otimes v^{(l)}_m,$$

where W_r is a certain submodule of F_r called q-Wakimoto module [10].

The operator $J^-(u)$ is a generating function of a part of generators of the Drinfeld realization for $U_q(\widehat{sl_2})$ at level k.

The operator $S(t)$ commutes with $U_q(\widehat{sl_2})$ modulo total differences. Here modulo total differences means modulo functions of the form

$${}_{k+2}\partial_z f(z) = \frac{f(q^{k+2}z) - f(q^{-(k+2)}z)}{(q-q^{-1})z}.$$

Consider

$$F(t,z) = \langle L + \sum_{i=1}^n l_i - 2N|\phi^{(l_1)}(z_1)\cdots\phi^{(l_n)}(z_n)S(t_N)\cdots S(t_1)|L\rangle$$

which is a function taking the value in $V^{(l_1)}\otimes\cdots\otimes V^{(l_n)}$. Let

$$\triangle_j = \frac{j(j+2)}{4(k+2)}.$$

Set

$$\widehat{F} = \left(\prod_{i=1}^n z_i^{\triangle_{L+\sum_{j=i}^n l_j - 2N} - \triangle_{L+\sum_{j=i+1}^n l_j - 2N}}\right) F = \left(\prod_{i=1}^n z_i^{\frac{l_i}{2(k+2)}\left(L+\sum_{i<j} l_j - 2N + \frac{l_i+2}{2}\right)}\right) F.$$

Then the function $\widehat{F}(t,z)$ satisfies qKZ equation (1) with $\kappa = q^{-2\left(L + \frac{\sum_{i=1}^n l_i}{2} - N + 1\right)}$ modulo total differences [10].

4 Integral formulae

Define the components of $F(t,z)$ by

$$F(t,z) = \sum_{\substack{\nu_i \in \{0,\dots,l_i\} \\ 1 \le i \le n}} F^{(\nu)}(t,z) v^{(l_1)}_{\nu_1} \otimes \cdots \otimes v^{(l_n)}_{\nu_n},$$

where $(\nu) = (\nu_1, \dots, \nu_n)$. By the conditions on weights $F^{(\nu)}(t,z) = 0$ unless

$$\sum_{i=1}^{n} (l_i - \nu_i) = N$$

is satisfied. We assume this condition once for all. Let

$$\begin{aligned} &\sharp\{i \,|\, \nu_i \neq l_i\} = r, \qquad \{i \,|\, \nu_i \neq l_i\} = \{k(1) < \cdots < k(r)\}, \\ &n_i = l_{k(i)} - \nu_{k(i)} \qquad (1 \le i \le r). \end{aligned}$$

The main result of this paper is

Theorem 1. *We have*

$$F^{(\nu)}(t,z) = A^{(\nu)}(t,z) \left(\prod_{i=1}^{n} z_i^{\frac{l_i}{2(k+2)}\left(L-2N+\sum\limits_{i<j} l_j\right)} \right) \left(\prod_{i<j} \xi_{l_i,l_j}(z_i/z_j) \right) \Phi(t,z) w_{(-\nu)}(t,z),$$

where $(-\nu) = (l_1 - \nu_1, \dots, l_n - \nu_n)$, $n_i = l_{k(i)} - \nu_{k(i)}$ *and*

$$\begin{aligned} A^{(\nu)}(t,z) &= q^{-NL} q^{\frac{3N(N-1)}{2} - \left(\sum\limits_{i=1}^{n} l_i\right) N} q^{\frac{1}{2(k+2)}\left(k \sum\limits_{i<j} l_i l_j + k(L-2N) \sum\limits_{i=1}^{n} l_i + 4LN - 4N(N-1)\right)} \\ &\times \left(\frac{1}{q-q^{-1}} \right)^N \sum_{(\nu)} \left\{ \prod_{s=1}^{r} q^{\left(\sum\limits_{t=s+1}^{r} n_t\right) n_s - l_{k(s)} n_s} \right\} \left\{ \prod_{s=1}^{r} \prod_{i=0}^{n_s-1} \left(1 - q^{2(l_{k(s)}-i)}\right) \right\} \\ &\times \left(\prod_{a=1}^{N} t_a^{\frac{2}{k+2}(a-1) - \frac{1}{k+2}L - 1} \right). \end{aligned}$$

The formula for $F^{(\nu)}(t,z)$ is of the form of (2), (3). More precisely in Tarasov–Varchenko's formula (2), (3), W can be written as

$$W = \left(\prod_{i=1}^{n} z_i^{\frac{l_i}{2(k+2)}\left(L-3N-\sum\limits_{j<i} l_j + \sum_{i<j} l_j\right)} \right) \left(\prod_{a=1}^{N} t_a \right) A^{(\nu)}(t,z) W'$$

for suitable W'. This W' specifies an intertwiner. In this paper we don't consider the problem on specifying W'.

To prove Theorem 1 let us begin by writing down the formula obtained by the free field description of operators $\phi_l(z)$, $J^-(u)$, $S(t)$ given in Appendix C. Let $(\epsilon) = (\epsilon_1, \dots, \epsilon_N), (\mu) = (\mu_{1,1}, \dots, \mu_{1,n_1}, \dots, \mu_{r,n_r}) \in \{0,1\}^N$. Then $F^{(\nu)}(t,z)$ can be written as

$$F^{(\nu)}(t,z) = (-1)^N \left(q - q^{-1}\right)^{-2N} \prod_{i=1}^{r} \frac{1}{[n_i]!} \prod_{a=1}^{N} t_a^{-1}$$

$$\times \sum_{\epsilon_i,\mu_{i_1,i_2}=\pm} \prod_{i=1}^N \epsilon_i \oint \left(\prod_{\substack{1\le i_1\le r\\ 1\le i_2\le n_{i_1}}} \mu_{i_1,i_2} \frac{du_{i_1,i_2}}{2\pi i u_{i_1,i_2}} \right) F^{(\nu)}_{(\epsilon)(\mu)}(t,z|u),$$

where

$$\begin{aligned} F^{(\nu)}_{(\epsilon)(\mu)}(t,z|u) &= \Big\langle L+\sum_{i=1}^n l_i - 2N | \phi_{l_1}(z_1)\cdots\phi_{l_{k(1)-1}}(z_{k(1)-1}) \\ &\times [\ldots[\phi_{l_{k(1)}}(z_{k(1)}), J^-_{\mu_{1,1}}(u_{1,1})]_{q^{l_{k(1)}}}, J^-_{\mu_{1,2}}(u_{1,2})]_{q^{l_{k(1)}-2}}\ldots, J^-_{\mu_{1,n_1}}(u_{1,n_1})]_{q^{l_{k(1)}-2(n_1-1)}}\cdots \\ &\times [\ldots[\phi_{l_{k(r)}}(z_{k(r)}), J^-_{\mu_{r,1}}(u_{r,1})]_{q^{l_{k(r)}}}, J^-_{\mu_{r,2}}(u_{r,2})]_{q^{l_{k(r)}-2}}\ldots, J^-_{\mu_{r,n_r}}(u_{r,n_r})]_{q^{l_{k(r)}-2(n_r-1)}} \\ &\times \phi_{l_{k(r)+1}}(z_{k(r)+1})\ldots\phi_{l_n}(z_n) S_{\epsilon_N}(t_N)\ldots S_{\epsilon_1}(t_1)|L\Big\rangle. \end{aligned}$$

and the integrand in the right hand side signifies to take the coefficient of $\left(\prod_{\substack{1\le i\le r\\ 1\le j\le n_i}} u_{i,j}\right)^{-1}$. For the notation $[x,y]_q$ see Appendix C.

Let $(m)=(m_1,\ldots,m_r)$, $0\le m_i\le n_i$. Then

$$\begin{aligned} &\oint \left(\prod_{\substack{1\le i_1\le r\\ 1\le i_2\le n_{i_1}}} \mu_{i_1,i_2} \frac{du_{i_1,i_2}}{2\pi i u_{i_1,i_2}} \right) F^{(\nu)}_{(\epsilon)(\mu)}(t,z) \\ &= \sum_{\substack{0\le m_i\le n_i\\ 1\le i\le r}} (-1)^{\sum_{i=1}^r m_i} \left(\prod_{i=1}^r q^{m_i l_{k(i)}} q^{-m_i(n_i-1)} \begin{bmatrix} n_i \\ m_i \end{bmatrix} \right) \\ &\times \int_{C^N} \left(\prod_{\substack{1\le i_1\le r\\ 1\le i_2\le n_{i_1}}} \mu_{i_1,i_2} \frac{du_{i_1,i_2}}{2\pi i u_{i_1,i_2}} \right) F^{(\nu)}_{(\epsilon)(\mu)(m)}(t,z|u), \end{aligned}$$

where

$$\begin{aligned} F^{(\nu)}_{(\epsilon)(\mu)(m)}(t,z|u) &= \Big\langle L+\sum_{i=1}^n l_i - 2N | \phi_{l_1}(z_1)\cdots\phi_{l_{k(1)-1}}(z_{k(1)-1}) \\ &\times \big(J^-_{\mu_{1,1}}(u_{1,1})\cdots J^-_{\mu_{1,m_1}}(u_{1,m_1})\phi_{l_{k(1)}}(z_{k(1)}) J^-_{\mu_{1,m_1+1}}(u_{1,m_1+1})\cdots J^-_{\mu_{1,n_1}}(u_{1,n_1})\big)\cdots \\ &\times \big(J^-_{\mu_{r,1}}(u_{r,1})\cdots J^-_{\mu_{r,m_r}}(u_{r,m_r})\phi_{l_{k(r)}}(z_{k(r)}) J^-_{\mu_{r,m_r+1}}(u_{r,m_r+1})\cdots J^-_{\mu_{r,n_r}}(u_{r,n_r})\big) \\ &\times \phi_{l_{k(r)+1}}(z_{k(r)+1})\cdots\phi_{l_n}(z_n) S_{\epsilon_N}(t_N)\cdots S_{\epsilon_1}(t_1)|L\Big\rangle, \end{aligned}$$

and C^N is a suitable deformation of the torus $\mathbb{T}^N$ specified as follows. We introduce the lexicographical order

$$(i_1,i_2)<(j_1,j_2) \quad \Leftrightarrow \quad i_1<j_1 \quad \text{or} \quad i_1=j_1 \quad \text{and} \quad i_2<j_2.$$

For a given $(m)=(m_1,\ldots,m_r)$, $1\le m_i\le n_i$, we define

$$\begin{aligned} j<(i_1,i_2) \quad &\Leftrightarrow \quad j<k(i_1) \quad \text{or} \quad j=k(i_1) \quad \text{and} \quad m_{i_1}<i_2, \\ j>(i_1,i_2) \quad &\Leftrightarrow \quad j>k(i_1) \quad \text{or} \quad j=k(i_1) \quad \text{and} \quad m_{i_1}\ge i_2. \end{aligned}$$

The contour for the integration variable u_{i_1,i_2} is a simple closed curve rounding the origin in the counterclockwise direction such that $q^{l_j+k+2}z_j$ $((i_1,i_2)<j)$, $q^{-2}u_{j_1,j_2}$ $((i_1,i_2)<(j_1,j_2))$, $q^{-\mu_{i_1,i_2}(k+2)}t_a$ $(1\le a\le N)$ are inside, and $q^{-l_j+k+2}z_j$ $((i_1,i_2)>j)$, $q^2u_{j_1,j_2}$ $((j_1,j_2)<(i_1,i_2))$ are outside. We denote it $C_{(i_1,i_2)}$.

Then

$$F^{(\nu)}_{(\epsilon)(\mu)(m)}(t,z|u)=f^{(\nu)}(t,z)\Phi(t,z)G^{(\nu)}_{(\epsilon)(\mu)(m)}(t,z|u),$$

where

$$\begin{aligned}
f^{(\nu)}(t,z)&=\left\{\prod_{i<j}(q^kz_i)^{\frac{l_il_j}{2(k+2)}}\xi_{l_i,l_j}(z_i/z_j)\right\}\left\{\prod_{i=1}^{n}(q^kz_i)^{-\frac{Nl_i}{k+2}}\right\}\\
&\quad\times\left\{\prod_{i=1}^{n}(q^kz_i)^{\frac{Ll_i}{2(k+2)}}\right\}\left\{\prod_{i=1}^{N}(q^{-2}t_i)^{-\frac{L}{k+2}}\right\}\left\{\prod_{a<b}(q^{-2}t_b)^{\frac{2}{k+2}}\right\},\\
G^{(\nu)}_{(\epsilon)(\mu)(m)}(t,z|u)&=\widehat{G}^{(\nu)}_{(\epsilon)(\mu)(m)}(t,z|u)\left(\prod_{a<b}\frac{q^{\epsilon_b}t_b-q^{\epsilon_a}t_a}{t_b-q^{-2}t_a}\right),\\
\widehat{G}^{(\nu)}_{(\epsilon)(\mu)(m)}(t,z|u)&=\left(\prod_{(i_1,i_2)}q^{L\mu_{i_1,i_2}}\right)\left(\prod_{(i_1,i_2)>j}\frac{z_j-q^{\mu_{i_1,i_2}l_j-k-2}u_{i_1,i_2}}{z_j-q^{l_j-k-2}u_{i_1,i_2}}\right)\\
&\quad\times\left(\prod_{(i_1,i_2)<j}q^{\mu_{i_1,i_2}l_j}\frac{u_{i_1,i_2}-q^{-\mu_{i_1,i_2}l_j+k+2}z_j}{u_{i_1,i_2}-q^{l_j+k+2}z_j}\right)\\
&\quad\times\left(\prod_{\substack{(i_1,i_2)\\1\le b\le N}}q^{-\mu_{i_1,i_2}}\frac{u_{i_1,i_2}-q^{-\mu_{i_1,i_2}(k+1)-\epsilon_b}t_b}{u_{i_1,i_2}-q^{-\mu_{i_1,i_2}(k+2)}t_b}\right)\\
&\quad\times\left(\prod_{(i_1,i_2)<(j_1,j_2)}\frac{q^{-\mu_{i_1,i_2}}u_{i_1,i_2}-q^{-\mu_{j_1,j_2}}u_{j_1,j_2}}{u_{i_1,i_2}-q^{-2}u_{j_1,j_2}}\right).
\end{aligned}$$

For i, let $A^{\pm}_{\mu,i}=\{(i,j)|\mu_{i,j}=\pm\}$. The number of elements in $A^{\pm}_{\mu,i}$ is $a^{\pm}_i$and $A^{\pm}_{\mu,i}=\{\ell^{\pm}_{i,1},\dots,\ell^{\pm}_{i,a^{\pm}_i}\}$. We set $a^-_i=a_i$, $A^-_{\mu,i}=A_{\mu,i}$, $A_\mu=\cup_{i=1}^{r}A_{\mu,i}$ and

$$\begin{aligned}
\widehat{J}^{(\nu)}_{(\epsilon)(\mu)}&=\sum_{\substack{0\le m_i\le n_i\\1\le i\le r}}(-1)^{\sum_{i}^{r}m_i}\left\{\prod_{i=1}^{r}q^{m_il_{k(i)}}q^{-m_i(n_i-1)}\begin{bmatrix}n_i\\m_i\end{bmatrix}\right\}\\
&\quad\times\int_{C^N}\left(\prod_{(i_1,i_2)}\mu_{i_1,i_2}\frac{du_{i_1,i_2}}{2\pi iu_{i_1,i_2}}\right)\widehat{G}^{(\nu)}_{(\epsilon)(\mu)(m)}.
\end{aligned}$$

See the beginning of the next section for the notation of the q-binomial coefficient $\begin{bmatrix}n_i\\m_i\end{bmatrix}$.

For a given $(a)=(a_1,\dots,a_r)$, $1\le a_i\le n_i$, we define $\widehat{J}^{(\nu)}_{(\epsilon)(a)}$ and $J^{(\nu)}_{(a)}$ as follows

$$\widehat{J}^{(\nu)}_{(\epsilon)(a)}=\sum_{\substack{|A_{\mu,i}|=a_i\\1\le i\le r}}\widehat{J}^{(\nu)}_{(\epsilon)(\mu)},$$

$$J_{(a)}^{(\nu)} = \sum_{\epsilon_1,\ldots,\epsilon_N=\pm} \left(\prod_{j=1}^{N} \epsilon_j \right) \left(\prod_{1\le a<b\le N} \frac{q^{\epsilon_b} t_b - q^{\epsilon_a} t_a}{t_b - q^{-2} t_a} \right) \widehat{J}_{(\epsilon)(a)}^{(\nu)}.$$

Using $J_{(a)}^{(\nu)}$, $F^{(\nu)}(t,z)$ can be written as

$$F^{(\nu)}(t,z) = (-1)^N \left(q - q^{-1}\right)^{-2N} \left(\prod_{i=1}^{r} \frac{1}{[n_i]!} \right) \left(\prod_{b=1}^{N} t_b^{-1} \right) f^{(\nu)}(t,z) \Phi(t,z) \sum_{(a)} J_{(a)}^{(\nu)}.$$

Theorem 1 straightforwardly follows from the following proposition.

Proposition 1. *If $(a) \neq (n_1, n_2, \ldots, n_r)$, $J_{(a)}^{(\nu)}(t,z) = 0$. For $(a) = (n_1, n_2, \ldots, n_r)$ we have*

$$J_{(n_1,\ldots,n_r)}^{(\nu)}(t,z) = (-1)^N \left(1 - q^{-2}\right)^N q^{N(N-L) + \frac{N(N-1)}{2} - \left(\sum_{i=1}^{n} l_i\right) N}$$
$$\times \prod_{s=1}^{r} \left\{ q^{\left(\sum_{t=s+1}^{r} n_t\right) n_s - l_{k(s)} n_s} [n_s]! \prod_{i=0}^{n_s - 1} \left(1 - q^{2(l_{k(s)} - i)}\right) \right\} w_{(-\nu)}(t,z).$$

This proposition is proved by performing integrals in the variables $u_{i,j}$ in the next section.

5 Proof of Proposition 1

We set

$$[n]! = \prod_{i=1}^{n} [i], \qquad \left[\begin{array}{c} n \\ m \end{array} \right] = \frac{[n]!}{[n-m]![m]!},$$

for nonnegative integers n, m $(n \geq m)$. To prove Proposition 1, we have to calculate $\widehat{J}_{(\epsilon)(a)}^{(\nu)}$. We need the following lemmas.

Lemma 1. *For $n \geq 1$ and $n \geq m \geq 0$, we have*

$$(i) \qquad \sum_{\substack{A \sqcup B = \{1,2,\ldots,n\} \\ |A| = m}} \left(\prod_{\substack{i<j \\ i \in A, j \in B}} q^2 \right) = q^{m(n-m)} \left[\begin{array}{c} n \\ m \end{array} \right];$$

$$(ii) \qquad \sum_{\substack{A \sqcup B = \{1,2,\ldots,n\} \\ |A| = m \\ \mu_i = 1 (i \in A),\, \mu_i = -1 (i \in B)}} \left(\prod_{i<j} q^{\mu_i} \right) = q^{-\frac{n(n-1)}{2} + m(n-1)} \left[\begin{array}{c} n \\ m \end{array} \right].$$

Proof. By the q-binomial theorem

$$\prod_{i=1}^{n} \left(1 + q^{-n-1+2i} x\right) = \sum_{i=0}^{n} \left[\begin{array}{c} n \\ i \end{array} \right] x^i,$$

we have the equation

$$\sum_{1 \le i_1 < \cdots < i_m \le n} q^{2 \sum_{j=1}^{m} i_j} = q^{(n+1)m} \left[\begin{array}{c} n \\ m \end{array} \right].$$

The assertions (i) and (ii) easily follow from this equation. ■

Lemma 2. *Let $n \geq 1$, $n \geq m \geq 0$ and $1 \leq i_1 < \cdots < i_m \leq n$. Then we have*

$$\sum_{\sigma \in S_n} \operatorname{sgn}\sigma \; t_{\sigma(i_1)} t_{\sigma(i_2)} \cdots t_{\sigma(i_m)} \prod_{1 \leq a < b \leq n} (t_{\sigma(b)} - q^{-2} t_{\sigma(a)})$$
$$= q^{-m(n+1) - \frac{n(n-1)}{2} + 2\sum_{j=1}^{m} i_j} [m]![n-m]! \; e_m(t_1, \ldots, t_n) \prod_{1 \leq a < b \leq n} (t_b - t_a),$$

where $e_m(t_1, \ldots, t_n)$ is the m-th elementary symmetric polynomial.

Proof. Set

$$F(t) = \sum_{\sigma \in S_n} \operatorname{sgn}\sigma \; t_{\sigma(i_1)} t_{\sigma(i_2)} \cdots t_{\sigma(i_m)} \prod_{1 \leq a < b \leq n} (t_{\sigma(b)} - q^{-2} t_{\sigma(a)}).$$

It is easy to see that $F(t)$ is an antisymmetric polynomial. So we can write

$$F(t) = S(t) \prod_{1 \leq a < b \leq n} (t_b - t_a),$$

where $S(t)$ is a symmetric polynomial. Moreover $S(t)$ is a homogeneous polynomial of degree m and $\deg_{t_i} S(t) = 1$ for all $i \in \{1, \ldots, n\}$. Hence we have

$$S(t) = c e_m(t)$$

for some constant c.

The number $(-1)^{\sum_{j=1}^{m} i_j + \frac{n(n-1)}{2} - \frac{m(m+1)}{2}} c$ is equal to the coefficient of

$$t_{i_1}^n t_{i_2}^{n-1} \cdots t_{i_m}^{n-m+1} t_1^{n-m-1} t_2^{n-m-2} \cdots t_{n-1}$$

in $F(t)$.

We can show

$$c = q^{-2nm + m(m-1) + 2\sum_{k=1}^{m} i_k} \left(q^{-m(m-1)} \sum_{\sigma \in S_m} q^{2\ell(\sigma)} \right) \left(q^{-(n-m)(n-m-1)} \sum_{\tau \in S_{n-m}} q^{2\ell(\tau)} \right),$$

where $\ell(\sigma)$ is the inversion number of σ.

Using the fact $\sum_{\sigma \in S_m} q^{2\ell(\sigma)} = q^{\frac{m(m-1)}{2}} [m]!$, we have the desired result. ■

Lemma 3. *For $1 \leq n \leq l$, we have*

$$\sum_{s=0}^{n} (-1)^s q^{-s(n-1)} \begin{bmatrix} n \\ s \end{bmatrix} \sum_{\sigma \in S_n} \prod_{i=1}^{s} (z - q^l t_{\sigma(i)}) \prod_{i=s+1}^{n} (z - q^{-l} t_{\sigma(i)}) \left(\prod_{1 \leq a < b \leq n} \frac{t_{\sigma(b)} - q^{-2} t_{\sigma(a)}}{t_{\sigma(b)} - t_{\sigma(a)}} \right)$$
$$= (-1)^n q^{-ln - \frac{n(n-1)}{2}} \left\{ \prod_{i=0}^{n-1} \left(1 - q^{2(l-i)}\right) \right\} [n]! t_1 t_2 \ldots t_n.$$

Proof. We set

$$L_{n,s} = \sum_{\sigma \in S_n} \operatorname{sgn}\sigma \prod_{i=1}^{s} (z - q^l t_{\sigma(i)}) \prod_{j=s+1}^{n} (z - q^{-l} t_{\sigma(j)}) \left(\prod_{i > j} \frac{t_{\sigma(i)} - q^{-2} t_{\sigma(j)}}{t_i - t_j} \right),$$

$$L_n = \sum_{s=0}^{n} (-1)^s q^{-s(n-1)} \begin{bmatrix} n \\ s \end{bmatrix} L_{n,s}.$$

Using Lemma 2,

$$\begin{aligned} L_{n,s} &= \sum_{k=0}^{n} (-1)^k z^{n-k} e_k(t) q^{-k(n+1)-\frac{n(n-1)}{2}} [k]![n-k]! \left\{ \sum_{t=0}^{k} q^{2lt-lk} \left(\sum_{\substack{1\le i_1<i_2<\cdots<i_t\le s \\ s<i_{t+1}<\cdots<i_k\le n}} q^{\sum_{j=1}^{k} 2i_j} \right) \right\} \\ &= \sum_{k=0}^{n} (-1)^k z^{n-k} e_k(t) q^{-lk-k(n+1)-\frac{n(n-1)}{2}} [k]![n-k]! \\ &\quad \times \left(\sum_{t=0}^{k} q^{2lt} q^{2s(k-t)+(s+1)t+(n-s+1)(k-t)} \begin{bmatrix} s \\ t \end{bmatrix} \begin{bmatrix} n-s \\ k-t \end{bmatrix} \right). \end{aligned}$$

Then,

$$\begin{aligned} L_n &= \sum_{s=0}^{n} (-1)^s q^{-s(n-1)} \begin{bmatrix} n \\ s \end{bmatrix} \sum_{k=0}^{n} (-1)^k z^{n-k} e_k(t) q^{-lk-k(n+1)-\frac{n(n-1)}{2}} [k]![n-k]! \\ &\quad \times \left(\sum_{t=0}^{k} q^{2lt} q^{sk+k+n(k-t)} \begin{bmatrix} s \\ t \end{bmatrix} \begin{bmatrix} n-s \\ k-t \end{bmatrix} \right) \\ &= [n]! \sum_{k=0}^{n} (-1)^k z^{n-k} e_k(t)\, q^{-lk-k(n+1)-\frac{n(n-1)}{2}} \\ &\quad \times \sum_{t=0}^{k} q^{2lt} q^{(k-t)(n+1)+t} \begin{bmatrix} k \\ t \end{bmatrix} \sum_{s=t}^{n-k+t} (-1)^s q^{-s(n-k-1)} \begin{bmatrix} n-k \\ s-t \end{bmatrix} \\ &= [n]! \sum_{k=0}^{n} (-1)^k z^{n-k} e_k(t)\, q^{-lk-k(n+1)-\frac{n(n-1)}{2}} \\ &\quad \times \sum_{t=0}^{k} q^{2lt} q^{(k-t)(n+1)+t} \begin{bmatrix} k \\ t \end{bmatrix} (-1)^t q^{-t(n-k-1)} \delta_{n,k} \\ &= [n]!(-1)^n q^{-ln} q^{-\frac{n(n-1)}{2}} \sum_{t=0}^{n} (-1)^t q^{2lt} q^{-(n-1)t} \begin{bmatrix} n \\ t \end{bmatrix} e_n(t) \\ &= [n]!(-1)^n q^{-ln} q^{-\frac{n(n-1)}{2}} \left\{ \prod_{i=0}^{n-1} (1-q^{2(l-i)}) \right\} e_n(t). \end{aligned}$$

Here we have used the q-binomial theorem. ■

For a given sequence $(m_i)_{i=1}^{r}$ $(0 \le m_i \le n_i)$, let $M_i = \{(i,j) \,|\, j \le m_i\}$. Set

$$\widehat{I}^{(\nu)}_{(\mu)(\epsilon)(m)} = \int_{C^N} \left(\prod_{(i_1,i_2)} \frac{du_{i_1,i_2}}{2\pi i u_{i_1,i_2}} \right) \widehat{G}^{(\nu)}_{(\mu)(\epsilon)(m)}.$$

Lemma 4. *We have*

$$\widehat{I}^{(\nu)}_{(\mu)(\epsilon)(m)} = q^{(L-N)\left\{\sum_{s=1}^{r}(n_s-2a_s)\right\}} \left(\prod_{(i_1,i_2)<j} q^{\mu_{i_1,i_2} l_j} \right) \left(\prod_{(i_1,i_2)<(j_1,j_2)} q^{-\mu_{i_1,i_2}} \right)$$

$$\times \sum_{\substack{C_i \sqcup D_i = A_{\mu,i} \\ D'_i = D_i \cap M_i \\ 1 \le i \le r}} \left(\prod_{b=1}^{N} q^{-1-\epsilon_b} \right)^{\sum_{i=1}^{r} |C_i|} \left(\prod_{\substack{(i_1,i_2)<(j_1,j_2) \\ (i_1,i_2) \in C_1 \cup \cdots \cup C_r \\ (j_1,j_2) \in D_1 \cup \cdots \cup D_r}} q^2 \right)$$

$$\times \sum_{\substack{1 \le b_{i,j} \le N \\ 1 \le i \le r \\ 1 \le j \le |D_i|}} \prod_{i_1=1}^{r} \left\{ \prod_{i_2=1}^{|D_{i_1}|} \left((1 - q^{-1-\epsilon_{b_{i_1,i_2}}}) \prod_{b \ne b_{i_1,i_2}} \frac{t_{b_{i_1,i_2}} - q^{-1-\epsilon_b} t_b}{t_{b_{i_1,i_2}} - t_b} \right.\right.$$

$$\left.\left. \times \prod_{(i_1,i_2)<(j_1,j_2)} \frac{t_{b_{i_1,i_2}} - t_{b_{j_1,j_2}}}{t_{b_{i_1,i_2}} - q^{-2} t_{b_{j_1,j_2}}} \prod_{j=1}^{k(i_1)-1} \frac{z_j - q^{-l_j} t_{b_{i_1,i_2}}}{z_j - q^{l_j} t_{b_{i_1,i_2}}} \right) \prod_{i_2=|D'_{i_1}|+1}^{|D_{i_1}|} \frac{z_{k(i_1)} - q^{-l_{k(i_1)}} t_{b_{i_1,i_2}}}{z_{k(i_1)} - q^{l_{k(i_1)}} t_{b_{i_1,i_2}}} \right\}.$$

Proof. We integrate with respect the variables $u_{i,j}$, $(i,j) \in A^+_\mu$ in the order $u_{\ell^+_{1,1}}, \dots, u_{\ell^+_{1,a^+_1}}$, $u_{\ell^+_{2,1}}, \dots, u_{\ell^+_{r,a^+_r}}$. With respect to $u_{\ell^+_{1,1}}$ the only singularity outside $C_{\ell^+_{1,1}}$ is ∞. Then the integral in $u_{\ell^+_{1,1}}$ is calculated by taking the residue at ∞. After this integration the integrand as a function of $u_{\ell^+_{1,2}}$ has a similar structure. Then the integral with respect to $u_{\ell^+_{1,2}}$ is calculated by taking residue at ∞ and so on. Finally we get

$$\widehat{I}^{(\nu)}_{(\epsilon)(\mu)(m)} = (-1)^{\sum_{i=1}^{r} a^+_i} \operatorname*{Res}_{u_{\ell^+_{r,a^+_r}}=\infty} \cdots \operatorname*{Res}_{u_{\ell^+_{r,1}}=\infty} \cdots \operatorname*{Res}_{u_{\ell^+_{1,a^+_1}}=\infty} \cdots \operatorname*{Res}_{u_{\ell^+_{1,1}}=\infty} \widehat{G}^{\nu}_{(\epsilon)(\mu)(m)}(t,z|u)$$

$$= \left(\prod_{(i_1,i_2)} q^{(L-N)\mu_{i_1,i_2}} \right) \left(\prod_{(i_1,i_2)<j} q^{\mu_{i_1,i_2} l_j} \right) \left(\prod_{(i_1,i_2)<(j_1,j_2)} q^{-\mu_{i_1,i_2}} \right)$$

$$\times \int_{C^{N-\sum_{i=1}^{r} a^+_i}} \left(\prod_{(i_1,i_2)\in A_\mu} \frac{du_{i_1,i_2}}{2\pi i u_{i_1,i_2}} \right) \left(\prod_{\substack{j<(i_1,i_2) \\ (i_1,i_2)\in A_\mu}} \frac{z_j - q^{-l_j-k-2} u_{i_1,i_2}}{z_j - q^{l_j-k-2} u_{i_1,i_2}} \right)$$

$$\times \left(\prod_{\substack{(i_1,i_2)\in A_\mu \\ 1 \le b \le N}} \frac{u_{i_1,i_2} - q^{k+1-\epsilon_b} t_b}{u_{i_1,i_2} - q^{k+2} t_b} \right) \left(\prod_{\substack{(i_1,i_2)<(j_1,j_2) \\ (i_1,i_2),(j_1,j_2)\in A_\mu}} \frac{u_{i_1,i_2} - u_{j_1,j_2}}{u_{i_1,i_2} - q^{-2} u_{j_1,j_2}} \right),$$

where $C^{N-\sum_{i=1}^{r} a^+_i}$ is the resulting contour for $(u_{\ell_{1,1}}, \dots, u_{\ell_{r,a_r}})$. We set

$$I^{(\nu)+}_{(\epsilon)(\mu)(m)}(t,z) = \left(\prod_{(i_1,i_2)\in A_\mu} \frac{1}{u_{i_1,i_2}} \right) \left(\prod_{\substack{j<(i_1,i_2) \\ (i_1,i_2)\in A_\mu}} \frac{z_j - q^{-l_j-k-2} u_{i_1,i_2}}{z_j - q^{l_j-k-2} u_{i_1,i_2}} \right)$$

$$\times \left(\prod_{\substack{(i_1,i_2)\in A_\mu \\ 1 \le b \le N}} \frac{u_{i_1,i_2} - q^{k+1-\epsilon_b} t_b}{u_{i_1,i_2} - q^{k+2} t_b} \right) \left(\prod_{\substack{(i_1,i_2)<(j_1,j_2) \\ (i_1,i_2),(j_1,j_2)\in A_\mu}} \frac{u_{i_1,i_2} - u_{j_1,j_2}}{u_{i_1,i_2} - q^{-2} u_{j_1,j_2}} \right).$$

Next we perform integrations with respect to the remaining variables $u_{i,j}$, $(i,j) \in A_\mu$ in the order $u_{\ell_{r,a_r}}, \dots, u_{\ell_{r,1}}, u_{\ell_{r-1,a_{r-1}}}, \dots, u_{\ell_{1,1}}$. The poles of the integrand inside $C_{\ell_{r,a_r}}$ are 0 and $q^{k+2}t_b$, $b = 1, \dots, N$. Thus we have

$$\begin{aligned}
&\int_{C_{\ell_{r,a_r}}} \frac{du_{\ell_{r,a_r}}}{2\pi i} I^{(\nu)+}_{(\epsilon)(\mu)(m)}(t,z) \\
&= \left(\prod_{i \le b \le N} q^{-1-\epsilon_b} \right) \left(\prod_{\substack{(i_1,i_2)\in A_\mu \\ (i_1,i_2)\neq \ell_{r,a_r}}} \frac{1}{u_{i_1,i_2}} \right) \left(\prod_{\substack{j<(i_1,i_2) \\ (i_1,i_2)\in A_\mu - \{\ell_{r,a_r}\}}} \frac{z_j - q^{-l_j-k-2} u_{i_1,i_2}}{z_j - q^{l_j-k-2} u_{i_1,i_2}} \right) \\
&\times \left(\prod_{\substack{(i_1,i_2)\in A_\mu - \{\ell_{r,a_r}\} \\ 1\le b \le N}} \frac{u_{i_1,i_2} - q^{k+1-\epsilon_b} t_b}{u_{i_1,i_2} - q^{k+2} t_b} \right) \left(\prod_{\substack{(i_1,i_2)<(j_1,j_2)<\ell_{r,a_r} \\ (i_1,i_2),(j_1,j_2)\in A_\mu}} \frac{u_{i_1,i_2} - u_{j_1,j_2}}{u_{i_1,i_2} - q^{-2} u_{j_1,j_2}} \right) \\
&+ \sum_{1 \le b_{\ell_{r,a_r}} \le N} (1 - q^{-1-\epsilon_{b_{\ell_{r,a_r}}}}) \left(\prod_{j<\ell_{r,a_r}} \frac{z_j - q^{-l_j} t_{b_{\ell_{r,a_r}}}}{z_j - q^{l_j-k-2} t_{b_{\ell_{r,a_r}}}} \right) \left(\prod_{\substack{1\le b \le N \\ b \neq b_{\ell_{r,a_r}}}} \frac{t_{b_{\ell_{r,a_r}}} - q^{-1-\epsilon_b} t_b}{t_{b_{\ell_{r,a_r}}} - t_b} \right) \\
&\times \left(\prod_{(i_1,i_2)<\ell_{r,a_r}} \frac{u_{i_1,i_2} - q^{k+2} t_{b_{\ell_{r,a_r}}}}{u_{i_1,i_2} - q^{k} t_{b_{\ell_{r,a_r}}}} \right) \left(\prod_{\substack{(i_1,i_2)\in A_\mu \\ (i_1,i_2)\neq \ell_{r,a_r}}} \frac{1}{u_{i_1,i_2}} \right) \left(\prod_{\substack{j<(i_1,i_2) \\ (i_1,i_2)\in A_\mu - \{\ell_{r,a_r}\}}} \frac{z_j - q^{-l_j-k-2} u_{i_1,i_2}}{z_j - q^{l_j-k-2} u_{i_1,i_2}} \right) \\
&\times \left(\prod_{\substack{(i_1,i_2)\in A_\mu - \{\ell_{r,a_r}\} \\ 1\le b \le N}} \frac{u_{i_1,i_2} - q^{k+1-\epsilon_b} t_b}{u_{i_1,i_2} - q^{k+2} t_b} \right) \left(\prod_{\substack{(i_1,i_2)<(j_1,j_2)<\ell_{r,a_r} \\ (i_1,i_2),(j_1,j_2)\in A_\mu}} \frac{u_{i_1,i_2} - u_{j_1,j_2}}{u_{i_1,i_2} - q^{-2} u_{j_1,j_2}} \right).
\end{aligned}$$

The integrand in $u_{\ell_{r,a_r-1}}$ has the poles at 0 and $q^{k+2}t_b$ inside $C_{\ell_{r,a_r-1}}$ and so on. Finally we get

$$\begin{aligned}
\widehat{I}^{(\nu)}_{(\epsilon)(\mu)} = &\left(\prod_{(i_1,i_2)} q^{(L-N)\mu_{i_1,i_2}} \right) \left(\prod_{(i_1,i_2)<j} q^{\mu_{i_1,i_2} l_j} \right) \left(\prod_{(i_1,i_2)<(j_1,j_2)} q^{-\mu_{i_1,i_2}} \right) \\
&\times \sum_{\substack{w_{\ell_{i_1,i_2}} \in \{0\}\cup(T - W_{i_1,i_2}) \\ (i_1,i_2)\in A_\mu}} \operatorname*{Res}_{u_{\ell_{1,1}} = w_{\ell_{1,1}}} \cdots \operatorname*{Res}_{u_{\ell_{r,a_r}} = w_{\ell_{r,a_r}}} I^{(\nu)+}_{(\epsilon)(\mu)},
\end{aligned}$$

where $T = \{t_1, t_2, \dots, t_N\}$, $W_{i_1,i_2} = \bigcup_{\ell_{i_1,i_2} < \ell_{j_1,j_2}} \{w_{\ell_{j_1,j_2}}\}$.

Set $C_i = \{\ell_{i,j} \,|\, w_{\ell_{i,j}} = 0\}$, $D_i = A_{\mu,i} - C_i$. Then we have the desired result. ■

Now we can calculate $\widehat{J}^{(\nu)}_{(\epsilon)(a)}$.

Proposition 2. *We have*

$$\widehat{J}^{(\nu)}_{(\epsilon)(a)} = (-1)^{\sum_{i=1}^r a_i} \left(q^{\sum_{s=1}^r \left(\sum_{t=k(s)+1}^n l_t \right)(n_s - 2a_s)} \right) \left(q^{(L-N)\left\{ \sum_{s=1}^r (n_s - 2a_s) \right\}} \right)$$

$$\times\left(q^{-\sum_{s=1}^{r}\sum_{t=s+1}^{r} n_s(n_t-2a_t)}\right)\sum_{\substack{1\le b_{i_1,i_2}\le N\\ 1\le i_1\le r\\ 1\le i_2\le a_{i_1}}}\left(\prod_{i_1<j_1}\frac{t_{b_{i_1,i_2}}-t_{b_{i_1,i_2}}}{t_{b_{i_1,i_2}}-q^{-2}t_{b_{i_1,i_2}}}\right)$$

$$\times\prod_{i_1=1}^{r}\left\{\sum_{s_{i_1}=0}^{a_{i_1}} q^{a_{i_1}(n_{i_1}-s_{i_1}-1)-\frac{n_{i_1}(n_{i_1}-1)}{2}}\frac{[n_{i_1}]!}{[s_{i_1}]![a_{i_1}-s_{i_1}]!}\right.$$

$$\times\sum_{i=0}^{n_{i_1}-a_{i_1}}(-1)^{i+s_{i_1}}q^{i(2l_{k(i_1)}-n_{i_1}-a_{i_1}+1)+s_{i_1}}\frac{1}{[i]![n_{i_1}-a_{i_1}-i]!}$$

$$\times\left\{\prod_{i_2=1}^{a_{i_1}}\left(\left(1-q^{-1-\epsilon_{b_{i_1,i_2}}}\right)\prod_{b\ne b_{i_1,i_2}}\frac{t_{b_{i_1,i_2}}-q^{-1-\epsilon_b}t_b}{t_{b_{i_1,i_2}}-t_b}\prod_{i_2<j_2}\frac{t_{b_{i_1,i_2}}-t_{b_{i_1,j_2}}}{t_{b_{i_1,i_2}}-q^{-2}t_{b_{i_1,j_2}}}\right.\right.$$

$$\left.\left.\left.\times\prod_{j=1}^{k(i_1)-1}\frac{z_j-q^{-l_j}t_{b_{i_1,i_2}}}{z_j-q^{l_j}t_{b_{i_1,i_2}}}\right)\prod_{i_2=s_{i_1}+1}^{a_{i_1}}\frac{z_{k(i_1)}-q^{-l_{k(i_1)}}t_{b_{i_1,i_2}}}{z_{k(i_1)}-q^{l_{k(i_1)}}t_{b_{i_1,i_2}}}\right\}\right\}.$$

Proof. Using Lemma 4 we have

$$\widehat{J}^{(\nu)}_{(\epsilon)(a)}=(-1)^{\sum_{i=1}^{r}a_i}\sum_{\substack{|A_{\mu,i}|=a_i\\ 1\le i\le r}}\sum_{\substack{0\le m_i\le n_i\\ 1\le i\le r}}(-1)^{\sum_{i=1}^{r}m_i}\left\{\prod_{i=1}^{r}q^{m_il_{k(i)}}q^{-m_i(n_i-1)}\begin{bmatrix} n_i\\ m_i\end{bmatrix}\right\}$$

$$\times\left(\prod_{i=1}^{r}q^{(L-N)(n_i-2a_i)}\right)\left(\prod_{(i_1,i_2)<j}q^{\mu_{i_1,i_2}l_j}\right)\left(\prod_{(i_1,i_2)<(j_1,j_2)}q^{-\mu_{i_1,i_2}}\right)$$

$$\times\sum_{\substack{C_i\sqcup D_i=A_{\mu,i}\\ D'_i=D_i\cap M_i\\ 1\le i\le r}}\Big(\prod_{b=1}^{N}q^{-1-\epsilon_b}\Big)^{\sum_{i=1}^{r}|C_i|}\left(\prod_{\substack{(i_1,i_2)<(j_1,j_2)\\ (i_1,i_2)\in C_1\cup\cdots\cup C_r\\ (j_1,j_2)\in D_1\cup\cdots\cup D_r}}q^2\right)$$

$$\times\sum_{\substack{1\le b_{i,j}\le N\\ 1\le i\le r\\ 1\le j\le |D_i|}}\prod_{i_1=1}^{r}\left\{\prod_{i_2=1}^{|D_{i_1}|}\left(\left(1-q^{-1-\epsilon_{b_{i_1,i_2}}}\right)\prod_{b\ne b_{i_1,i_2}}\frac{t_{b_{i_1,i_2}}-q^{-1-\epsilon_b}t_b}{t_{b_{i_1,i_2}}-t_b}\right.\right.$$

$$\left.\times\prod_{(i_1,i_2)<(j_1,j_2)}\frac{t_{b_{i_1,i_2}}-t_{b_{j_1,j_2}}}{t_{b_{i_1,i_2}}-q^{-2}t_{b_{j_1,j_2}}}\prod_{j=1}^{k(i_1)-1}\frac{z_j-q^{-l_j}t_{b_{i_1,i_2}}}{z_j-q^{l_j}t_{b_{i_1,i_2}}}\right)$$

$$\left.\times\prod_{i_2=|D'_{i_1}|+1}^{|D_{i_1}|}\frac{z_{k(i_1)}-q^{-l_{k(i_1)}}t_{b_{i_1,i_2}}}{z_{k(i_1)}-q^{l_{k(i_1)}}t_{b_{i_1,i_2}}}\right\}. \tag{4}$$

Set $\lambda_i=|A_{\mu,i}\cap M_i|$, $\gamma_i=|D_i|$, $s_i=|D'_i|$, $1\le i\le r$. Then the right hand side of (4) is equal to

$$\sum_{\substack{0\le m_i\le n_i\\ 1\le i\le r}}(-1)^{\sum_{i=1}^{r}m_i}\left\{\prod_{i=1}^{r}q^{m_il_{k(i)}}q^{-m_i(n_i-1)}\begin{bmatrix} n_i\\ m_i\end{bmatrix}\right\}$$

$$\times\sum_{\substack{0\le\gamma_j\le a_j\\ 1\le j\le r}}\sum_{\substack{0\le s_j\le\gamma_j\\ 1\le j\le r}}\sum_{\substack{0\le\lambda_j\le m_j\\ 1\le j\le r}}C_{(a)(\gamma)}\left\{q^{\sum_{s=1}^{r}l_{k(s)}(m_s-2\lambda_s)}\right.$$

$$\times \left\{ \prod_{i_1=1}^{r} \left(\sum_{\substack{|A_{\mu,i_1}|=a_{i_1} \\ |A_{\mu,i_1}\cap M_{i_1}|=\lambda_{i_1}}} \prod_{\substack{i_2<j_2 \\ i_1=j_1}} q^{-\mu_{i_1,i_2}} \right) \right\}$$

$$\times \left(\prod_{i_1=1}^{r} q^{\lambda_{i_1}\gamma_{i_1}+a_{i_1}\gamma_{i_1}-a_{i_1}s_{i_1}-\gamma_{i_1}^2} \left[\begin{array}{c} \lambda_{i_1} \\ s_{i_1} \end{array} \right] \left[\begin{array}{c} a_{i_1}-\lambda_{i_1} \\ \gamma_{i_1}-s_{i_1} \end{array} \right] \right)$$

$$\times \sum_{\substack{1\le b_{i_1,i_2}\le N \\ 1\le i_1\le r \\ 1\le i_2\le \gamma_{i_1}}} \prod_{i_1=1}^{r} \left\{ \left(\prod_{i_2=1}^{\gamma_{i_1}} (1-q^{-1-\epsilon_{b_{i_1,i_2}}}) \prod_{b\neq b_{i_1,i_2}} \frac{t_{b_{i_1,i_2}}-q^{-1-\epsilon_b}t_b}{t_{b_{i_1,i_2}}-t_b} \right.\right.$$

$$\left.\left. \times \prod_{i_2<j_2} \frac{t_{b_{i_1,i_2}}-t_{b_{i_1,j_2}}}{t_{b_{i_1,i_2}}-q^{-2}t_{b_{i_1,j_2}}} \prod_{j=1}^{k(i_1)-1} \frac{z_j-q^{-l_j}t_{b_{i_1,i_2}}}{z_j-q^{l_j}t_{b_{i_1,i_2}}} \right) \prod_{i_2=s_{i_1}+1}^{\gamma_{i_1}} \frac{z_{k(i_1)}-q^{-l_{k(i_1)}}t_{b_{i_1,i_2}}}{z_{k(i_1)}-q^{l_{k(i_1)}}t_{b_{i_1,i_2}}} \right\}$$

$$\times \left(\prod_{i_1<j_1} \frac{t_{b_{i_1,i_2}}-t_{b_{j_1,j_2}}}{t_{b_{i_1,i_2}}-q^{-2}t_{b_{j_1,j_2}}} \right),$$

where

$$C_{(a)(\gamma)} = (-1)^{\sum_{i=1}^{r} a_i} \left(q^{\sum_{s=1}^{r}\left(\sum_{t=k(s)+1}^{n} l_t\right)(n_s-2a_s)} \right) \left(q^{(L-N)\left\{\sum_{s=1}^{r}(n_s-2a_s)\right\}} \right)$$

$$\times \left(q^{-\sum_{s=1}^{r}(n_s-2a_s)\left(\sum_{t=s+1}^{r} n_t\right)} \right) \left(q^{2\sum_{s=1}^{r}\sum_{s<t}\gamma_t(a_s-\gamma_s)} \right) \left(\prod_{1\le b\le N} q^{-1-\epsilon_b} \right)^{\sum_{s=1}^{r}(a_s-\gamma_s)}.$$

Here we have used Lemma 1 (i).

By (ii) of Lemma 1 we have

$$\widehat{J}^{(\nu)}_{(\epsilon)(a)} = \sum_{j=1}^{r} \sum_{\substack{0\le\gamma_j\le a_j \\ 1\le j\le r}} C_{(a)(\gamma)} \sum_{\substack{1\le b_{i_1,i_2}\le N \\ 1\le i_1\le r \\ 1\le i_2\le\gamma_{i_2}}} \left(\prod_{i_1<j_1} \frac{t_{b_{i_1,i_2}}-t_{b_{j_1,j_2}}}{t_{b_{i_1,i_2}}-q^{-2}t_{b_{j_1,j_2}}} \right)$$

$$\times \prod_{i_1=1}^{r} \left\{ \sum_{s_{i_1}=0}^{\gamma_{i_1}} \sum_{\lambda_{i_1}=s_{i_1}}^{a_j-\gamma_{i_1}+s_{i_1}} \sum_{m_{i_1}=0}^{n_{i_1}} (-1)^{m_{i_1}} q^{-m_{i_1}(n_{i_1}-1)} q^{2l_{k(i_1)}(m_{i_1}-\lambda_{i_1})} \left[\begin{array}{c} n_{i_1} \\ m_{i_1} \end{array} \right] \right.$$

$$\times \left(\sum_{\substack{|A_{\mu,i_1}|=a_{i_1} \\ |A_{\mu,i_1}\cap M_{i_1}|=\lambda_{i_1}}} \prod_{\substack{i_2<j_2 \\ i_1=j_1}} q^{-\mu_{i_1,i_2}} \right) \left(q^{\lambda_{i_1}\gamma_{i_1}+a_{i_1}\gamma_{i_1}-a_{i_1}s_{i_1}-\gamma_{i_1}^2} \left[\begin{array}{c} \lambda_{i_1} \\ s_{i_1} \end{array} \right] \left[\begin{array}{c} a_{i_1}-\lambda_{i_1} \\ \gamma_{i_1}-s_{i_1} \end{array} \right] \right)$$

$$\times \left\{ \left(\prod_{i_2=1}^{\gamma_{i_1}} (1-q^{-1-\epsilon_{b_{i_1,i_2}}}) \prod_{b\neq b_{i_1,i_2}} \frac{t_{b_{i_1,i_2}}-q^{-1-\epsilon_b}t_b}{t_{b_{i_1,i_2}}-t_b} \prod_{i_2<j_2} \frac{t_{b_{i_1,i_2}}-t_{b_{i_1,j_2}}}{t_{b_{i_1,i_2}}-q^{-2}t_{b_{i_1,j_2}}} \right.\right.$$

$$\left.\left.\left. \times \prod_{j=1}^{k(i_1)-1} \frac{z_j-q^{-l_j}t_{b_{i_1,i_2}}}{z_j-q^{l_j}t_{b_{i_1,i_2}}} \right) \prod_{i_2=s_{i_1}+1}^{\gamma_{i_1}} \frac{z_{k(i_1)}-q^{-l_{k(i_1)}}t_{b_{i_1,i_2}}}{z_{k(i_1)}-q^{l_{k(i_1)}}t_{b_{i_1,i_2}}} \right\} \right\}$$

$$
\begin{aligned}
&= \sum_{\substack{0\le\gamma_j\le a_j\\1\le j\le r}} C_{(a)(\gamma)} \sum_{\substack{1\le b_{i_1,i_2}\le N\\1\le i_1\le r\\1\le i_2\le\gamma_{i_2}}} \left(\prod_{i_1<j_1}\frac{t_{b_{i_1,i_2}}-t_{b_{j_1,j_2}}}{t_{b_{i_1,i_2}}-q^{-2}t_{b_{j_1,j_2}}}\right)\\
&\times\prod_{i_1=1}^{r}\left\{\sum_{s_{i_1}=0}^{\gamma_{i_1}}\sum_{\lambda_{i_1}=s_{i_1}}^{a_j-\gamma_{i_1}+s_{i_1}}\sum_{m_{i_1}=0}^{n_{i_1}}(-1)^{m_{i_1}}q^{-m_{i_1}(n_{i_1}-1)}q^{2l_{k(i_1)}(m_{i_1}-\lambda_{i_1})}\begin{bmatrix}n_{i_1}\\m_{i_1}\end{bmatrix}\right.\\
&\times\left(q^{n_{i_1}\lambda_{i_1}+a_{i_1}n_{i_1}-a_{i_1}m_{i_1}-a_{i_1}-\frac{n_{i_1}(n_{i_1}-1)}{2}}\begin{bmatrix}m_{i_1}\\\lambda_{i_1}\end{bmatrix}\begin{bmatrix}n_{i_1}-m_{i_1}\\a_{i_1}-\lambda_{i_1}\end{bmatrix}\right)\\
&\times\left(q^{\lambda_{i_1}\gamma_{i_1}+a_{i_1}\gamma_{i_1}-a_{i_1}s_{i_1}-\gamma_{i_1}^2}\begin{bmatrix}\lambda_{i_1}\\s_{i_1}\end{bmatrix}\begin{bmatrix}a_{i_1}-\lambda_{i_1}\\\gamma_{i_1}-s_{i_1}\end{bmatrix}\right)\\
&\times\left\{\prod_{i_2=1}^{\gamma_{i_1}}\left((1-q^{-1-\epsilon_{b_{i_1,i_2}}})\prod_{b\ne b_{i_1,i_2}}\frac{t_{b_{i_1,i_2}}-q^{-1-\epsilon_b}t_b}{t_{b_{i_1,i_2}}-t_b}\prod_{i_2<j_2}\frac{t_{b_{i_1,i_2}}-t_{b_{i_1,j_2}}}{t_{b_{i_1,i_2}}-q^{-2}t_{b_{i_1,j_2}}}\right.\right.\\
&\left.\left.\left.\times\prod_{j=1}^{k(i_1)-1}\frac{z_j-q^{-l_j}t_{b_{i_1,i_2}}}{z_j-q^{l_j}t_{b_{i_1,i_2}}}\right)\prod_{i_2=s_{i_1}+1}^{\gamma_{i_1}}\frac{z_{k(i_1)}-q^{-l_{k(i_1)}}t_{b_{i_1,i_2}}}{z_{k(i_1)}-q^{l_{k(i_1)}}t_{b_{i_1,i_2}}}\right\}\right\}.
\end{aligned}
$$

It is easy to show

$$
\begin{aligned}
&\sum_{\lambda=s}^{a-\gamma+s}\sum_{m=0}^{n}(-1)^m q^{-m(n-1)}\begin{bmatrix}n\\m\end{bmatrix}q^{2l(m-\lambda)}\left(q^{n\lambda+an-am-a-\frac{n(n-1)}{2}}\begin{bmatrix}m\\\lambda\end{bmatrix}\begin{bmatrix}n-m\\a-\lambda\end{bmatrix}\right)\\
&\quad\times\left(q^{\lambda\gamma+a\gamma-as-\gamma^2}\begin{bmatrix}\lambda\\s\end{bmatrix}\begin{bmatrix}a-\lambda\\\gamma-s\end{bmatrix}\right)\\
&\quad=(-1)^s q^{a(n-s-1)+s}q^{-\frac{n(n-1)}{2}}\frac{[n]!}{[s]![a-s]!}\sum_{i=0}^{n-a}(-1)^i q^{i(2l-n-a+1)}\frac{1}{[i]![n-a-i]!}\delta_{a,\gamma},
\end{aligned}
$$

for $0\le s\le\gamma\le a\le n$.

Hence

$$
\begin{aligned}
\widehat{J}^{(\nu)}_{(\epsilon)(a)}&=(-1)^{\sum_{i=1}^{r}a_i}\left(q^{\sum_{s=1}^{r}\left(\sum_{t=k(s)+1}^{n}l_t\right)(n_s-2a_s)}\right)\left(q^{(L-N)\left\{\sum_{s=1}^{r}(n_s-2a_s)\right\}}\right)\\
&\times\left(q^{-\sum_{s=1}^{r}(n_s-2a_s)\left(\sum_{t=s+1}^{r}n_t\right)}\right)\sum_{\substack{1\le b_{i_1,i_2}\le N\\1\le i_1\le r\\1\le i_2\le a_{i_1}}}\left(\prod_{i_1<j_1}\frac{t_{b_{i_1,i_2}}-t_{b_{j_1,j_2}}}{t_{b_{i_1,i_2}}-q^{-2}t_{b_{j_1,j_2}}}\right)\\
&\times\prod_{i_1=1}^{r}\left\{\sum_{s_{i_1}=0}^{a_{i_1}}(-1)^{s_{i_1}}q^{a_{i_1}(n_{i_1}-s_{i_1}-1)+s_{i_1}}q^{-\frac{n_{i_1}(n_{i_1}-1)}{2}}\frac{[n_{i_1}]!}{[s_{i_1}]![a_{i_1}-s_{i_1}]!}\right.\\
&\times\sum_{i=0}^{n_{i_1}-a_{i_1}}(-1)^i q^{i(2l_{k(i_1)}-n_{i_1}-a_{i_1}+1)}\frac{1}{[i]![n_{i_1}-a_{i_1}-i]!}\\
&\times\left\{\prod_{i_2=1}^{a_{i_1}}\left((1-q^{-1-\epsilon_{b_{i_1,i_2}}})\prod_{b\ne b_{i_1,i_2}}\frac{t_{b_{i_1,i_2}}-q^{-1-\epsilon_b}t_b}{t_{b_{i_1,i_2}}-t_b}\prod_{i_2<j_2}\frac{t_{b_{i_1,i_2}}-t_{b_{i_1,j_2}}}{t_{b_{i_1,i_2}}-q^{-2}t_{b_{i_1,j_2}}}\right.\right.\\
&\left.\left.\left.\times\prod_{j=1}^{k(i_1)-1}\frac{z_j-q^{-l_j}t_{b_{i_1,i_2}}}{z_j-q^{l_j}t_{b_{i_1,i_2}}}\right)\prod_{i_2=s_{i_1}+1}^{a_{i_1}}\frac{z_{k(i_1)}-q^{-l_{k(i_1)}}t_{b_{i_1,i_2}}}{z_{k(i_1)}-q^{l_{k(i_1)}}t_{b_{i_1,i_2}}}\right\}\right\}.
\end{aligned}
$$

■

Lemma 5. *If $a_i \neq n_i$ for some i,*

$$\sum_{\substack{\epsilon_i=\pm\\1\le i\le N}}\left(\prod_{j=1}^N \epsilon_j\right)\left(\prod_{a<b}\frac{q^{\epsilon_b}t_b-q^{\epsilon_a}t_a}{t_b-q^{-2}t_a}\right)\widehat{J}^{(\nu)}_{(\epsilon)(a)}=0.$$

Proof. It is enough to show the following equation. For $1\le b_{i_1,i_2}\le N$ $(1\le i_1\le r,\ 1\le i_2\le a_{i_1})$, $b_{i_1,i_2}\neq b_{j_1,j_2}$ $((i_1,i_2)\neq(j_1,j_2))$,

$$\sum_{\substack{\epsilon_i=\pm\\1\le i\le N}}\left(\prod_{i=1}^N \epsilon_i\right)\prod_{a<b}(q^{\epsilon_b}t_b-q^{\epsilon_a}t_a)\prod_{\substack{1\le i_1\le r\\1\le i_2\le a_{i_1}}}\left\{(1-q^{-1-\epsilon_{b_{i_1,i_2}}})\prod_{b\neq b_{i_1,i_2}}(t_{b_{i_1,i_2}}-q^{-1-\epsilon_b}t_b)\right\}$$
$$=(1-q^{-2})^N q^{\frac{N(N-1)}{2}}\left(\prod_{s=1}^r\delta_{a_s,n_s}\right)\left\{\prod_{a<b}(t_b-t_a)\right\}\left\{\prod_{b\neq b_{i_1,i_2}}(t_{b_{i_1,i_2}}-q^{-2}t_b)\right\}. \tag{5}$$

For a set $\{b_{1,1},\ldots,b_{r,a_r}\}=\{b_1,\ldots,b_\alpha\}$, let$\{c_1,\ldots,c_{N-\alpha}\}$ be defined by

$$\{b_1,\ldots,b_\alpha\}\sqcup\{c_1,\ldots,c_{N-\alpha}\}=\{1,\ldots,N\},$$

where $\alpha=\sum\limits_{i=1}^r a_i$.

Then the left hand side of (5) is equal to

$$(1-q^{-2})^\alpha\left(\prod_{1\le i\le\alpha}\delta_{\epsilon_{b_i},+}\right)\left\{\prod_{i<j}q(t_{b_j}-t_{b_i})\right\}\left\{\prod_{\substack{1\le i,j\le\alpha\\i\neq j}}(t_{b_i}-q^{-2}t_{b_j})\right\}$$
$$\times\left\{\prod_{b_i<c_j}(-q)\right\}\left\{\prod_{c_i<b_j}q\right\}\sum_{\substack{\epsilon_{c_i}=\pm\\1\le i\le N-\alpha}}\left(\prod_{i=1}^{N-\alpha}\epsilon_{c_i}\right)\left\{\prod_{i<j}(q^{\epsilon_{c_j}}t_{c_j}-q^{\epsilon_{c_i}}t_{c_i})\right\}$$
$$\times\left\{\prod_{1\le i\le\alpha}\prod_{1\le j\le N-\alpha}(t_{b_i}-q^{\epsilon_{c_j}-1}t_{c_j})\right\}\left\{\prod_{1\le i\le\alpha}\prod_{1\le j\le N-\alpha}(t_{b_i}-q^{-1-\epsilon_{c_j}}t_{c_j})\right\}.$$

Using

$$(t_{b_j}-q^{\epsilon_{c_i}-1}t_{c_i})(t_{b_i}-q^{-1-\epsilon_{c_j}}t_{c_j})=(t_{b_j}-t_{c_i})(t_{b_i}-q^{-2}t_{c_j}),$$

we have

$$\sum_{\epsilon}\left(\prod_{i=1}^N\epsilon_i\right)\left\{\prod_{a<b}(q^{\epsilon_b}t_b-q^{\epsilon_a}t_a)\right\}\left\{\prod_{\substack{1\le i_1\le r\\1\le i_2\le a_{i_1}}}(1-q^{-1-\epsilon_{b_{i_1,i_2}}})\right\}\left\{\prod_{b\neq b_{i_1,i_2}}(t_{b_{i_1,i_2}}-q^{-1-\epsilon_b}t_b)\right\}$$
$$=(1-q^{-2})^\alpha\left(\prod_{1\le i\le\alpha}\delta_{\epsilon_{b_i},+}\right)\left\{\prod_{i<j}q(t_{b_j}-t_{b_i})\right\}\left\{\prod_{\substack{1\le i,j\le\alpha\\i\neq j}}(t_{b_i}-q^{-2}t_{b_j})\right\}$$
$$\times\left\{\prod_{1\le i\le\alpha}\prod_{1\le j\le N-\alpha}(t_{b_i}-t_{c_j})(t_{b_i}-q^{-2}t_{c_j})\right\}$$

$$\times \left\{ \prod_{b_i<c_j} (-q) \right\} \left\{ \prod_{c_i<b_j} q \right\} \sum_{\substack{\epsilon_{c_i}=\pm \\ 1\le i\le N-\alpha}} \left(\prod_{i=1}^{N-\alpha} \epsilon_{c_i} \right) \left\{ \prod_{i<j} \left(q^{\epsilon_{c_j}} t_{c_j} - q^{\epsilon_{c_i}} t_{c_i} \right) \right\}.$$

Let $\alpha \neq N$ and $\boldsymbol{a}_i(\epsilon) = {}^t (1, q^\epsilon t_i, (q^\epsilon t_i)^2, \ldots, (q^\epsilon t_i)^{N-\alpha-1})$. Then

$$\begin{aligned} &\sum_{\substack{\epsilon_i=\pm \\ 1\le i\le N-\alpha}} \left(\prod_{i=1}^{N-\alpha} \epsilon_i \right) \prod_{i<j} (q^{\epsilon_j} t_j - q^{\epsilon_i} t_i) \\ &\quad = \sum_{\substack{\epsilon_i=\pm \\ 1\le i\le N-\alpha}} \left(\prod_{i=1}^{N-\alpha} \epsilon_i \right) \det(\boldsymbol{a}_1(\epsilon_1), \boldsymbol{a}_2(\epsilon_2), \ldots, \boldsymbol{a}_{N-\alpha}(\epsilon_{N-\alpha})). \end{aligned} \tag{6}$$

Since

$$\sum_{\epsilon_i=\pm} \epsilon_i \boldsymbol{a}_i(\epsilon) = {}^t (0, (q-q^{-1})t_i, \ldots, (q^{N-\alpha-1} - q^{-(N-\alpha-1)}) t_i^{N-\alpha-1}),$$

the right hand side of (6) is equal to 0. ∎

If $a_i = n_i$ for all i, then

$$\begin{aligned} &\sum_{\substack{\epsilon_i=\pm \\ 1\le i\le N}} \left(\prod_{i=1}^{N} \epsilon_i \right) \left(\prod_{1\le a<b\le N} \frac{q^{\epsilon_b} t_b - q^{\epsilon_a} t_a}{t_b - q^{-2} t_a} \right) \hat{J}^{(\nu)}_{(\epsilon)(a)} \\ &\quad = C_1 \left(\prod_{a<b} \frac{t_b - t_a}{t_b - q^{-2} t_a} \right) \sum_{\substack{\Gamma_1 \sqcup \cdots \sqcup \Gamma_r = \{1,\ldots,N\} \\ |\Gamma_s| = n_s \ (s=1,\ldots,r)}} \sum_{\substack{b_{i_1,i_2} \in \Gamma_{i_1} \\ 1\le i_1\le r \\ 1\le i_2\le n_{i_1}}} \left(\prod_{i_1>j_1} \frac{t_{b_{i_1,i_2}} - q^{-2} t_{b_{j_1,j_2}}}{t_{b_{i_1,i_2}} - t_{b_{j_1,j_2}}} \right) \\ &\quad \times \prod_{i_1=1}^{r} \left\{ \sum_{s_{i_1}=0}^{n_{i_1}} (-1)^{s_{i_1}} q^{-(n_{i_1}-1)s_{i_1}} \begin{bmatrix} n_{i_1} \\ s_{i_1} \end{bmatrix} \prod_{i_2=1}^{s_{i_1}} (z_{k(i_1)} - q^{l_{k(i_1)}} t_{b_{i_1,i_2}}) \right. \\ &\quad \times \prod_{i_2=s_{i_1}+1}^{n_{i_1}} (z_{k(i_1)} - q^{-l_{k(i_1)}} t_{b_{i_1,i_2}}) \prod_{i_2>j_2} \frac{t_{b_{i_1,i_2}} - q^{-2} t_{b_{i_1,j_2}}}{t_{b_{i_1,i_2}} - t_{b_{i_1,j_2}}} \\ &\quad \left. \times \prod_{i_2=1}^{n_{i_1}} \left(\frac{1}{z_{k(i_1)} - q^{l_{k(i_1)}} t_{b_{i_1,i_2}}} \prod_{j=1}^{k(i_1)-1} \frac{z_j - q^{-l_j} t_{b_{i_1,i_2}}}{z_j - q^{l_j} t_{b_{i_1,i_2}}} \right) \right\}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} C_1 &= (-1)^N \left(1 - q^{-2}\right)^N q^{N^2 - LN} q^{\frac{N(N-1)}{2}} q^{\sum_{i=1}^{r} \frac{n_i(n_i-1)}{2}} \\ &\quad \times \left\{ q^{-\sum_{s=1}^{r} \left(\sum_{t=k(s)+1}^{n} l_t \right) n_s} \right\} \left\{ q^{\sum_{s=1}^{r} \left(\sum_{t=s+1}^{r} n_t \right) n_s} \right\}. \end{aligned}$$

By Lemma 3 the right hand side of (7) becomes

$$C_1 \prod_{s=1}^{r} \left\{ (-1)^{n_s} [n_s]! q^{-l_{k(s)} n_s} q^{-\frac{n_s(n_s-1)}{2}} \left\{ \prod_{i=0}^{n_s-1} \left(1 - q^{2(l_{k(s)}-i)}\right) \right\} \right\}$$

$$\times \left(\prod_{a<b} \frac{t_b - t_a}{t_b - q^{-2}t_a}\right) \sum_{\substack{\Gamma_1 \sqcup \cdots \sqcup \Gamma_r = \{1,\ldots,N\} \\ |\Gamma_s| = n_s \ (s=1,\ldots,r)}} \left(\prod_{\substack{1 \le i < j \le r \\ a \in \Gamma_i, b \in \Gamma_j}} \frac{t_b - q^{-2}t_a}{t_b - t_a} \right)$$
$$\times \prod_{s=1}^{r} \prod_{b \in \Gamma_s} \left(\frac{t_b}{z_{k(s)} - q^{l_{k(s)}} t_b} \prod_{i=1}^{k(s)-1} \frac{z_i - q^{-l_i} t_b}{z_i - q^{l_i} t_b} \right).$$

This completes the proof of Proposition 1. ∎

A The representation $V_z^{(l)}$

Let q^{h_i}, e_i, f_i $(i = 0, 1)$ and q^d be the generators of $U_q(\widehat{sl_2})$. (See [4] for more details.) The actions of the generators of $U_q(\widehat{sl_2})$ on $V_z^{(l)}$ are given as follows.

For $0 \le i \le l$ and $n \in \mathbb{Z}$,

$$\begin{aligned}
&e_0 v_j^{(l)} \otimes z^n = [l-i] v_{i+1}^{(l)} \otimes z^{n+1}, && e_1 v_j^{(l)} \otimes z^n = [i] v_{i-1}^{(l)} \otimes z^n, \\
&f_0 v_j^{(l)} \otimes z^n = [i] v_{i-1}^{(l)} \otimes z^{n-1}, && f_1 v_j^{(l)} \otimes z^n = [l-i] v_{i+1}^{(l)} \otimes z^n, \\
&q^{h_0} v_i^{(l)} \otimes z^n = q^{-(l-2i)} v_i^{(l)} \otimes z^n, && q^{h_1} v_i^{(l)} \otimes z^n = q^{l-2i} v_i^{(l)} \otimes z^n, \\
&q^d v_j^{(l)} \otimes z^n = q^n v_j^{(l)} \otimes z^n.
\end{aligned}$$

B R-matrix

We give examples of explicit forms of R-matrix in the case of $l_1 = 1$ or $l_2 = 1$. They are taken from [4]. If we write

$$R_{1,l_2}(z)\big(v_\epsilon^{(1)} \otimes v_j^{(l_2)}\big) = \sum_{\epsilon'=0,1} v_{\epsilon'}^{(1)} \otimes r_{\epsilon'\epsilon}^{1 l_2}(z) v_j^{(l_2)},$$
$$R_{l_1,1}(z)\big(v_j^{(l_1)} \otimes v_\epsilon^{(1)}\big) = \sum_{\epsilon'=0,1} r_{\epsilon'\epsilon}^{l_1 1}(z) v_j^{(l_1)} \otimes v_{\epsilon'}^{(1)},$$

then we have

$$\begin{pmatrix} r_{00}^{1l_2}(z) & r_{01}^{1l_2}(z) \\ r_{10}^{1l_2}(z) & r_{11}^{1l_2}(z) \end{pmatrix} = \frac{1}{q^{1+l_2/2} - z^{-1}q^{-l_2/2}} \begin{pmatrix} q^{1+h/2} - z^{-1}q^{-h/2} & (q-q^{-1})z^{-1} f q^{h/2} \\ (q-q^{-1}) e q^{-h/2} & q^{1-h/2} - z^{-1}q^{h/2} \end{pmatrix},$$
$$\begin{pmatrix} r_{00}^{l_1 1}(z) & r_{01}^{l_1 1}(z) \\ r_{10}^{l_1 1}(z) & r_{11}^{l_1 1}(z) \end{pmatrix} = \frac{1}{zq^{l_1/2} - q^{-1-l_1/2}} \begin{pmatrix} zq^{h/2} - q^{-1-h/2} & (q-q^{-1}) z q^{h/2} f \\ (q-q^{-1}) q^{-h/2} e & zq^{-h/2} - q^{-1+h/2} \end{pmatrix},$$

$h = h_1$, $e = e_1$ and $f = f_1$.

C Free field representations

The following formulae are given in [6]. For $x = a, b, c$ let

$$x(L; M, N|z:\alpha) = -\sum_{n \ne 0} \frac{[Ln] x_n}{[Mn][Nn]} z^{-n} q^{|n|\alpha} + \frac{L\tilde{x}_0}{MN} \log z + \frac{L}{MN} Q_x,$$
$$x(N|z:\alpha) = x(L; L, N|z:\alpha) = -\sum_{n \ne 0} \frac{x_n}{[Nn]} z^{-n} q^{|n|\alpha} + \frac{\tilde{x}_0}{N} \log z + \frac{1}{N} Q_x.$$

The normal ordering is defined by specifying N_+, $\tilde{a}_0$, $\tilde{b}_0$, $\tilde{c}_0$ as annihilation operators, N_-, Q_a, Q_b, Q_c as creation operators.

Define operators

$$J^-(z):\ F_{r,s}\to F_{r,s+1},\qquad S(z):\ F_{r,s}\to F_{r-2,s-1},\qquad \phi_m^{(l)}(z):\ F_{r,s}\to F_{r+l,s+l-m},$$

by

$$\begin{aligned}
J^-(z) &= \frac{1}{(q-q^{-1})z}(J_+^-(z)-J_-^-(z)),\\
J_\mu^-(z) &=: \exp\left(a^{(\mu)}\left(q^{-2}z;-\frac{k+2}{2}\right)+b\big(2|q^{(\mu-1)(k+2)}z;-1\big)+c\big(2|q^{(\mu-1)(k+1)-1}z;0\big)\right):,\\
a^{(\mu)}\left(q^{-2}z;-\frac{k+2}{2}\right) &= \mu\left\{(q-q^{-1})\sum_{n=1}^{\infty}a_{\mu n}z^{-\mu n}q^{(2\mu-\frac{k+2}{2})n}+\tilde{a}_0\log q\right\},\\
S(z) &= \frac{-1}{(q-q^{-1})z}(S_+(z)-S_-(z)),\\
S_\epsilon(z) &=: \exp\left(-a\left(k+2|q^{-2}z;-\frac{k+2}{2}\right)-b\big(2|q^{-k-2}z;-1\big)-c\big(2|q^{-k-2+\epsilon}z;0\big)\right):,\\
\phi_l^{(l)}(z) &=: \exp\left(a\left(l;2,k+2|q^kz;\frac{k+2}{2}\right)\right):,\\
\phi_{l-r}^{(l)}(z) &= \frac{1}{[r]!}\oint\left(\prod_{j=1}^{r}\frac{du_j}{2\pi i}\right)\left[\cdots\left[\left[\phi_l^{(l)}(z),J^-(u_1)\right]_{q^l},J^-(u_2)\right]_{q^{l-2}},\dots,J^-(u_r)\right]_{q^{l-2r+2}},
\end{aligned}$$

where

$$[r]!=\prod_{i=1}^{r}[i],\qquad [X,Y]_q=XY-qYX,$$

and the integral in $\phi_{l-r}^{(l)}(z)$ signifies to take the coefficient of $(u_1\cdots u_r)^{-1}$.

D List of OPE's

The following formulae are given in [6]

$$\begin{aligned}
\phi_{l_1}(z_1)\phi_{l_2}(z_2) &= (q^kz_1)^{\frac{l_1l_2}{2(k+2)}}\frac{\left(q^{l_1+l_2+2k+6}\frac{z_2}{z_1};q^4,q^{2(k+2)}\right)_\infty\left(q^{-l_1-l_2+2k+6}\frac{z_2}{z_1};q^4,q^{2(k+2)}\right)_\infty}{\left(q^{l_1-l_2+2k+6}\frac{z_2}{z_1};q^4,q^{2(k+2)}\right)_\infty\left(q^{-l_1+l_2+2k+6}\frac{z_2}{z_1};q^4,q^{2(k+2)}\right)_\infty}\\
&\quad\times:\phi_{l_1}(z_1)\phi_{l_2}(z_2):,\qquad |q^{-l_1-l_2+2k+6}z_2|<|z_1|,\\
\phi_l(z)J_\mu^-(u) &= \frac{z-q^{\mu l-k-2}u}{z-q^{l-k-2}u}:\phi_l(z)J_\mu^-(u):,\qquad |q^{-l-k-2}u|<|z|,\\
J_\mu^-(u)\phi_l(z) &= q^{\mu l}\frac{u-q^{-\mu l+k+2}z}{u-q^{l+k+2}z}:\phi_l(z)J_\mu^-(u):,\qquad |q^{-l+k+2}u|<|z|,\\
\phi_l(z)S_\epsilon(t) &= \frac{(q^l\frac{t}{z};p)_\infty}{(q^{-l}\frac{t}{z};p)_\infty}(q^kz)^{-\frac{l}{k+2}}:\phi_l(z)S_\epsilon(t):,\qquad |z|>|q^{-l}t|,\\
J_\mu^-(u)S_\epsilon(t) &= q^{-\mu}\frac{u-q^{-\mu(k+1)-\epsilon}t}{u-q^{-\mu(k+2)}t}:J_\mu^-(u)S_\epsilon(t):,\qquad |u|>|q^{-k-2}t|,
\end{aligned}$$

$$J^-_{\mu_1}(u_1)J^-_{\mu_2}(u_2) = \frac{q^{-\mu_1}u_1 - q^{-\mu_2}u_2}{u_1 - q^{-2}u_2} : J^-_{\mu_1}(u_1)J^-_{\mu_2}(u_2) :, \qquad |u_1| > |q^{-2}u_2|,$$

$$S_{\epsilon_1}(t_1)S_{\epsilon_2}(t_2) = (q^{-2}t_1)^{\frac{2}{k+2}} \frac{q^{\epsilon_1}t_1 - q^{\epsilon_2}t_2}{t_1 - q^{-2}t_2} \frac{\left(q^{-2}\frac{t_2}{t_1}; p\right)_\infty}{\left(q^2\frac{t_2}{t_1}; p\right)_\infty} : S_{\epsilon_1}(t_1)S_{\epsilon_2}(t_2) :, \qquad |t_1| > |q^{-2}t_2|.$$

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