Symmetry, Integrability and Geometry: Methods and Applications SIGMA **5** (2009), 041, 14 pages

A First Order q-Difference System for the BC_1-Type Jackson Integral and Its Applications*

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Received December 01, 2008, in final form March 18, 2009; Published online April 03, 2009
doi:10.3842/SIGMA.2009.041

Abstract. We present an explicit expression for the q-difference system, which the BC_1-type Jackson integral (q-series) satisfies, as first order simultaneous q-difference equations with a concrete basis. As an application, we give a simple proof for the hypergeometric summation formula introduced by Gustafson and the product formula of the q-integral introduced by Nassrallah–Rahman and Gustafson.

Key words: q-difference equations; Jackson integral of type BC_1; Gustafson's C_n-type sum; Nassrallah–Rahman integral

2000 Mathematics Subject Classification: 33D15; 33D67; 39A13

1 Introduction

A lot of summation and transformation formulae for basic hypergeometric series have been found to date. The BC_1-type Jackson integral, which is the main subject of interest in this paper, is a q-series which can be written as a basic hypergeometric series in a class of so called *very-well-poised-balanced* ${}_{2r}\psi_{2r}$. A key reason to consider the BC_1-type Jackson integrals is to give an explanation of these hypergeometric series from the view points of the Weyl group symmetry and the q-difference equations of the BC_1-type Jackson integrals. In [15], we showed that Slater's transformation formula for a very-well-poised-balanced ${}_{2r}\psi_{2r}$ series can be regarded as a connection formula for the solutions of q-difference equations of the BC_1-type Jackson integral, i.e., the Jackson integral as a general solution of q-difference system is written as a linear combination of particular solutions. As a consequence we gave a simple proof of Slater's transformation formula. (See [15] for details. Also see [13] for a connection formula for the BC_n-type Jackson integral, which is a multisum generalization of that of type BC_1.)

The aim of this paper is to present an explicit form of the q-difference system as first order simultaneous q-difference equations for the BC_1-type Jackson integral with generic condition on the parameters. We give the Gauss decomposition of the coefficient matrix of the system with a concrete basis (see Theorem 4.1). Each entry of the decomposed matrices is written as a product of binomials and, as a consequence, the determinant of the coefficient matrix is easy to calculate explicitly. As an application we give a simple proof of the product formula for Gustafson's multiple C_n-type sum [10]. We also present an explicit form of the q-difference system for the BC_1-type Jackson integral with a balancing condition on the parameters. We finally give a simple proof of the product formula for the q-integral of Nassrallah–Rahman [16] and Gustafson [9]. A recent work of Rains and Spiridonov [17] contains results for the elliptic

*This paper is a contribution to the Proceedings of the Workshop "Elliptic Integrable Systems, Isomonodromy Problems, and Hypergeometric Functions" (July 21–25, 2008, MPIM, Bonn, Germany). The full collection is available at http://www.emis.de/journals/SIGMA/Elliptic-Integrable-Systems.html hypergeometric integral of a similar type to those contained for the BC_1-type Jackson integral obtained here.

2 BC_1-type Jackson integral

Throughout this paper, we assume $0 < q < 1$ and denote the *q-shifted factorial* for all integers N by $(x)_\infty := \prod_{i=0}^{\infty}(1 - q^i x)$ and $(x)_N := (x; q)_\infty/(q^N x; q)_\infty$.

Let $\mathcal{O}(\mathbb{C}^*)$ be the set of holomorphic functions on the complex multiplicative group $\mathbb{C}^*$. A function f on $\mathbb{C}^*$ is said to be *symmetric* or *skew-symmetric* under the Weyl group action $z \to z^{-1}$ if f satisfies $f(z) = f(z^{-1})$ or $f(z) = -f(z^{-1})$, respectively. For $\xi \in \mathbb{C}^*$ and a function f on $\mathbb{C}^*$, we define the sum over the lattice $\mathbb{Z}$

$$\int_0^{\xi\infty} f(z)\frac{d_q z}{z} := (1 - q)\sum_{\nu=-\infty}^{\infty} f(q^\nu \xi),$$

which, provided the integral converges, we call the *Jackson integral*. For an arbitrary positive integer s, we define the function Φ and the skew-symmetric function Δ on $\mathbb{C}^*$ as follows:

$$\Phi(z) := \prod_{m=1}^{2s+2} z^{\frac{1}{2}-\alpha_m}\frac{(qz/a_m)_\infty}{(za_m)_\infty}, \qquad \Delta(z) := z^{-1} - z, \tag{2.1}$$

where $a_m = q^{\alpha_m}$. For a symmetric function φ on $\mathbb{C}^*$ and a point $\xi \in \mathbb{C}^*$, we define the following sum over the lattice $\mathbb{Z}$:

$$\int_0^{\xi\infty} \varphi(z)\Phi(z)\Delta(z)\frac{d_q z}{z},$$

which we call the *Jackson integral of type BC_1* and is simply denoted by $\langle\varphi, \xi\rangle$. By definition the sum $\langle\varphi, \xi\rangle$ is invariant under the shift $\xi \to q^\nu\xi$ for $\nu \in \mathbb{Z}$.

Let $\Theta(z)$ be the function on $\mathbb{C}^*$ defined by

$$\Theta(z) := \frac{z^{s-\alpha_1-\cdots-\alpha_{2s+2}}\theta(z^2)}{\prod_{m=1}^{2s+2}\theta(a_m z)},$$

where $\theta(z)$ denotes the function $(z)_\infty(q/z)_\infty$, which satisfies

$$\theta(qz) = -\theta(z)/z \qquad \text{and} \qquad \theta(q/z) = \theta(z). \tag{2.2}$$

For a symmetric function $\varphi \in \mathcal{O}(\mathbb{C}^*)$, we denote the function $\langle\varphi, z\rangle/\Theta(z)$ by $\langle\langle\varphi, z\rangle\rangle$. We call $\langle\langle\varphi, z\rangle\rangle$ the *regularized Jackson integral of type BC_1*, which satisfies the following:

Lemma 2.1. *Assume $\alpha_1 + \alpha_2 + \cdots + \alpha_{2s+2} \notin \frac{1}{2} + \mathbb{Z}$. If $\varphi \in \mathcal{O}(\mathbb{C}^*)$ is symmetric, then the function $\langle\langle\varphi, z\rangle\rangle$ is symmetric and holomorphic on $\mathbb{C}^*$.*

Proof. See [15, Proposition 2.2]. ■

For an arbitrary meromorphic function φ on $\mathbb{C}^*$ we define the function $\nabla\varphi$ on $\mathbb{C}^*$ by

$$\nabla\varphi(z) := \varphi(z) - \frac{\Phi(qz)}{\Phi(z)}\varphi(qz).$$

In particular, from (2.1), the function $\Phi(qz)/\Phi(z)$ is the rational function

$$\frac{\Phi(qz)}{\Phi(z)} = q^{s+1}\prod_{m=1}^{2s+2}\frac{1-a_m z}{a_m - qz}.$$

The following proposition will be used for the proof of the key equation (Theorem 3.1):

Lemma 2.2. *If* $\int_0^{\xi\infty}\Phi(z)\varphi(z)\frac{d_qz}{z}$ *is convergent for* $\varphi \in \mathcal{O}(\mathbb{C}^*)$*, then* $\int_0^{\xi\infty}\Phi(z)\nabla\varphi(z)\frac{d_qz}{z} = 0$.

Proof. See [11, Lemma 5.1] for instance. ■

3 Key equation

In this section, we will present a key equation to construct the difference equations for the BC_1-type Jackson integral. Before we state it, we introduce the function $e(x;y)$ defined by

$$e(x;y) := x + x^{-1} - \left(y + y^{-1}\right),$$

which is expressed by the product form

$$e(x;y) = \frac{(y-x)(1-xy)}{xy}.$$

The basic properties of $e(x;y)$ are the following:

- $e(x;z) = e(x;y) + e(y;z),$ (3.1)
- $e(x;y) = -e(y;x), \quad e(x;y) = e\left(x^{-1};y\right),$ (3.2)
- $e(x;y)e(z;w) - e(x;z)e(y;w) + e(x;w)e(y;z) = 0.$ (3.3)

Remark 3.1. As we will see later, equation (3.3) is ignorable in the case $a_1a_2\cdots a_{2s+2} \neq 1$, while equation (3.1) is ignorable in the case $a_1a_2\cdots a_{2s+2} = 1$.

For functions f, g on $\mathbb{C}^*$, the function fg on $\mathbb{C}^*$ is defined by

$$(fg)(z) := f(z)g(z) \qquad \text{for} \qquad z \in \mathbb{C}^*.$$

Set $e_i(z) := e(z;a_i)$ and $(e_{i_1}e_{i_2}\cdots e_{i_s})(z) := e_{i_1}(z)e_{i_2}(z)\cdots e_{i_s}(z)$. The symbol $(e_{i_1}\cdots\widehat{e}_{i_k}\cdots e_{i_s})(z)$ is equal to $(e_{i_1}\cdots e_{i_{k-1}}e_{i_{k+1}}\cdots e_{i_s})(z)$. The key equation is the following:

Theorem 3.1. *Suppose* $a_i \neq a_j$ *if* $i \neq j$*. If* $\{i_1, i_2, \dots, i_s\} \subset \{1, 2, \dots, 2s+2\}$*, then*

$$C_0\langle e_{i_1}e_{i_2}\cdots e_{i_s}, \xi\rangle + \sum_{k=1}^{s} C_{i_k}\langle e_{i_1}\cdots\widehat{e}_{i_k}\cdots e_{i_s}, \xi\rangle = 0,$$

where the coefficients C_0 *and* C_{i_k} $(1 \le k \le s)$ *are given by*

$$C_0 = 1 - a_1a_2\cdots a_{2s+2} \qquad \text{and} \qquad C_{i_k} = \frac{\prod\limits_{m=1}^{2s+2}(1-a_{i_k}a_m)}{a_{i_k}^s(1-a_{i_k}^2)\prod\limits_{\substack{1\le\ell\le s\\ \ell\neq k}} e(a_{i_k};a_{i_\ell})}.$$

Proof. Without loss of generality, it suffices to show that

$$C_0\langle e_1e_2\cdots e_s,\xi\rangle+\sum_{i=1}^{s}C_i\,\langle e_1\cdots\widehat{e}_i\cdots e_s,\xi\rangle=0, \tag{3.4}$$

where the coefficients C_0 and C_i are given by

$$C_0=1-a_1a_2\cdots a_{2s+2}\qquad\text{and}\qquad C_i=\frac{\prod\limits_{m=1}^{2s+2}(1-a_ia_m)}{a_i^s(1-a_i^2)\prod\limits_{\substack{1\le k\le s\\ k\ne i}}e(a_i;a_k)}. \tag{3.5}$$

Set $F(z)=\prod_{m=1}^{2s+2}(a_m-z)$ and $G(z)=\prod_{m=1}^{2s+2}(1-a_mz)$. Then, from Lemma 2.2, it follows that

$$\int_0^{\xi\infty}\Phi(z)\nabla\left(\frac{F(z)}{z^{s+1}}\right)\frac{d_qz}{z}=0,\qquad\text{where}\qquad\nabla\left(\frac{F(z)}{z^{s+1}}\right)=\frac{F(z)-G(z)}{z^{s+1}}. \tag{3.6}$$

Since $(F(z)-G(z))/z^{s+1}$ is skew-symmetric under the reflection $z\to z^{-1}$, it is divisible by $z-z^{-1}$, and we can expand it as

$$\frac{F(z)-G(z)}{z^{s+1}(z-z^{-1})}=C_0\,e(z;a_1)e(z;a_2)\cdots e(z;a_s)+\sum_{i=1}^{s}C_i\,e(z;a_1)\cdots\widehat{e}(z;a_i)\cdots e(z;a_s), \tag{3.7}$$

where the coefficients C_i will be determined below. We obtain $C_0=1-a_1a_2\cdots a_{2s+2}$ from the principal term of asymptotic behavior of (3.7) as $z\to+\infty$. If we put $z=a_i$ $(1\le i\le s)$, then we have

$$-\frac{F(a_i)-G(a_i)}{a_i^s(1-a_i^2)}=C_i\prod_{\substack{1\le k\le s\\ k\ne i}}e(a_i;a_k).$$

Since $F(a_i)=0$ and $G(a_i)=\prod_{m=1}^{2s+2}(1-a_ia_m)$ by definition, the above equation implies (3.5). From (3.6) and (3.7), we obtain (3.4), which completes the proof. ■

4 The case $a_1a_2\cdots a_{2s+2}\ne 1$

4.1 q-difference equation

Set

$$v_k(z):=\begin{cases} e_{i_1}e_{i_2}\cdots e_{i_{s-1}}(z) & \text{if}\quad k=0,\\ e_{i_1}\cdots\widehat{e}_{i_k}\cdots e_{i_{s-1}}(z) & \text{if}\quad 1\le k\le s-1,\end{cases} \tag{4.1}$$

where the hat symbol denotes the term to be omitted.

Let T_{a_j} be the difference operator corresponding to the q-shift $a_j\to qa_j$.

Theorem 4.1. *Suppose $a_1a_2\cdots a_{2s+2}\ne 1$. For the BC_1-type Jackson integrals, if $\{i_1,i_2,\ldots,i_{s-1}\}\subset\{1,2,\ldots,2s+2\}$ and $j\notin\{i_1,i_2,\ldots,i_{s-1}\}$, then the first order vector-valued q-difference equation with respect to the basis $\{v_0,v_1,\ldots,v_{s-1}\}$ defined by* (4.1) *is given by*

$$T_{a_j}(\langle v_0,\xi\rangle,\ldots,\langle v_{s-1},\xi\rangle)=(\langle v_0,\xi\rangle,\ldots,\langle v_{s-1},\xi\rangle)B, \tag{4.2}$$

where $B = UL$. Here U and L are the $s \times s$ matrices defined by

$$U = \begin{pmatrix} c_0 & 1 & 1 & \cdots & 1 \\ & c_1 & & & \\ & & c_2 & & \\ & & & \ddots & \\ & & & & c_{s-1} \end{pmatrix}, \qquad L = \begin{pmatrix} 1 & & & & \\ d_1 & 1 & & & \\ d_2 & & 1 & & \\ \vdots & & & \ddots & \\ d_{s-1} & & & & 1 \end{pmatrix},$$

where

$$c_0 = \frac{\prod_{m=1}^{2s+2}(1 - a_j a_m)}{(-a_j)^s(1 - a_1 a_2 \cdots a_{2s+2})\bigl(1 - a_j^2\bigr) \prod_{\ell=1}^{s-1} e(a_{i_\ell}; a_j)}$$

and

$$c_k = e(a_{i_k}; a_j), \qquad d_k = \frac{\prod_{m=1}^{2s+2}(1 - a_{i_k} a_m)}{(-a_{i_k})^s(1-a_1 a_2 \cdots a_{2s+2})(1-a_{i_k}^2)e(a_j; a_{i_k}) \prod_{\substack{1\le\ell\le s-1 \\ \ell\neq k}} e(a_{i_\ell}; a_{i_k})} \tag{4.3}$$

for $k = 1, 2, \ldots, s-1$. Moreover,

$$\det B = \frac{\prod_{m=1}^{2s+2}(1 - a_j a_m)}{(-a_j)^s(1 - a_1 a_2 \cdots a_{2s+2})\bigl(1 - a_j^2\bigr)}.$$

Proof. Equation (4.2) is rewritten as $T_{a_j}(\langle v_0, \xi\rangle, \ldots, \langle v_{s-1}, \xi\rangle)L^{-1} = (\langle v_0, \xi\rangle, \ldots, \langle v_{s-1}, \xi\rangle)U$, where

$$L^{-1} = \begin{pmatrix} 1 & & & & \\ -d_1 & 1 & & & \\ -d_2 & & 1 & & \\ \vdots & & & \ddots & \\ -d_{s-1} & & & & 1 \end{pmatrix}.$$

Since $T_{a_j}\langle v_i, \xi\rangle = \langle e_j v_i, \xi\rangle$, the above equation is equivalent to

$$\langle e_j v_0, \xi\rangle - \sum_{k=1}^{s-1} d_k \langle e_j v_k, \xi\rangle = c_0 \langle v_0, \xi\rangle \tag{4.4}$$

and

$$\langle e_j v_k, \xi\rangle = \langle v_0, \xi\rangle + c_k \langle v_k, \xi\rangle \qquad \text{for} \qquad k = 1, 2, \ldots, s-1, \tag{4.5}$$

which are to be proved. Equation (4.4) is a direct consequence of (3.2) and Theorem 3.1 if $a_1 a_2 \cdots a_{2s+2} \neq 1$. Equation (4.5) is trivial using $e(z; a_j) = e(z; a_{i_k}) + e(a_{i_k}; a_j)$ from (3.1). Lastly $\det B = \det U \det L = c_0 c_1 \cdots c_{s-1}$, which completes the proof. ∎

Since the function $\Theta(z)$ satisfies $T_{a_j}\Theta(z) = -a_j \Theta(z)$, we immediately have the following from Theorem 4.1:

Corollary 4.1. *Suppose $a_1a_2\cdots a_{2s+2}\neq 1$. For the regularized BC_1-type Jackson integrals, if $\{i_1,i_2,\dots,i_{s-1}\}\subset\{1,2,\dots,2s+2\}$ and $j\notin\{i_1,i_2,\dots,i_{s-1}\}$, then the first order vector-valued q-difference equation with respect to the basis $\{v_0,v_1,\dots,v_{s-1}\}$ is given by*

$$T_{a_j}(\langle\langle v_0,\xi\rangle\rangle,\dots,\langle\langle v_{s-1},\xi\rangle\rangle)=(\langle\langle v_0,\xi\rangle\rangle,\dots,\langle\langle v_{s-1},\xi\rangle\rangle)\bar{B}, \tag{4.6}$$

where

$$\bar{B}=\frac{-1}{a_j}\begin{pmatrix} c_0 & 1 & 1 & \cdots & 1\\ & c_1 & & & \\ & & c_2 & & \\ & & & \ddots & \\ & & & & c_{s-1}\end{pmatrix}\begin{pmatrix} 1 & & & & \\ d_1 & 1 & & & \\ d_2 & & 1 & & \\ \vdots & & & \ddots & \\ d_{s-1} & & & & 1\end{pmatrix}$$

and c_i and d_i are given by (4.3). In particular, the diagonal entries of the upper triangular part are written as

$$-\frac{c_0}{a_j}=\frac{\prod_{m=1}^{2s+2}\left(1-a_j^{-1}a_m^{-1}\right)}{\left(1-a_1^{-1}a_2^{-1}\cdots a_{2s+2}^{-1}\right)\left(1-a_j^{-2}\right)\prod_{\ell=1}^{s-1}\left(1-a_{i_\ell}a_j^{-1}\right)\left(1-a_{i_\ell}^{-1}a_j^{-1}\right)}$$

and

$$-\frac{c_k}{a_j}=\left(1-a_{i_k}a_j^{-1}\right)\left(1-a_{i_k}^{-1}a_j^{-1}\right)\qquad \text{for}\qquad i=1,2,\dots,s-1.$$

Moreover,

$$\det\bar{B}=\frac{\prod_{m=1}^{2s+2}\left(1-a_j^{-1}a_m^{-1}\right)}{\left(1-a_j^{-2}\right)\left(1-a_1^{-1}a_2^{-1}\cdots a_{2s+2}^{-1}\right)}. \tag{4.7}$$

Remark 4.1. The q-difference system for the BC_n-type Jackson integral is discussed in [3] for its rank, and in [4, 5] for the explicit expression of the determinant of the coefficient matrix of the system. On the other hand, though it is only for the BC_1-type Jackson integral, the coefficient matrix in its Gauss decomposition form is obtained explicitly only in the present paper.

4.2 Application

The aim of this subsection is to give a simple proof of Gustafson's multiple C_n-type summation formula (Corollary 4.2). The point of the proof is to obtain a recurrence relation of Gustafson's multiple series of C_n-type. Before we state the recurrence relation, we first give the definition of the multiple series of C_n-type $\langle\langle 1,x\rangle\rangle_{\mathrm{G}}$.

For $z=(z_1,\dots,z_n)\in(\mathbb{C}^*)^n$, we set

$$\Phi_{\mathrm{G}}(z):=\prod_{i=1}^{n}\prod_{m=1}^{2s+2}z_i^{1/2-\alpha_m}\frac{(qa_m^{-1}z_i)_\infty}{(a_mz_i)_\infty},$$

$$\Delta_{C_n}(z):=\prod_{i=1}^{n}\frac{1-z_i^2}{z_i}\prod_{1\le j<k\le n}\frac{(1-z_j/z_k)(1-z_jz_k)}{z_j},$$

where $q^{\alpha_m} = a_m$. For an arbitrary $\xi = (\xi_1, \dots, \xi_n) \in (\mathbb{C}^*)^n$, we define the q-shift $\xi \to q^\nu \xi$ by a lattice point $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$, where $q^\nu \xi := (q^{\nu_1}\xi_1, \dots, q^{\nu_n}\xi_n) \in (\mathbb{C}^*)^n$. For $\xi = (\xi_1, \dots, \xi_n) \in (\mathbb{C}^*)^n$ we define the sum over the lattice $\mathbb{Z}^n$ by

$$\langle 1, \xi \rangle_{\rm G} := (1-q)^n \sum_{\nu \in \mathbb{Z}^n} \Phi_{\rm G}(q^\nu \xi) \Delta_{C_n}(q^\nu \xi),$$

which we call the BC_n*-type Jackson integral*. Moreover we set $\langle\!\langle 1, \xi \rangle\!\rangle_{\rm G} := \langle 1, \xi \rangle_{\rm G} / \Theta_{\rm G}(\xi)$, where

$$\Theta_{\rm G}(\xi) := \prod_{i=1}^{n} \frac{\xi_i^{i-\alpha_1-\alpha_2-\cdots-\alpha_{2s+2}} \theta(\xi_i^2)}{\prod\limits_{m=1}^{2s+2} \theta(a_m \xi_i)} \prod_{1 \le j < k \le n} \theta(\xi_j/\xi_k)\theta(\xi_j \xi_k).$$

By definition, it can be confirmed that $\langle\!\langle 1, \xi \rangle\!\rangle_{\rm G}$ is holomorphic on $(\mathbb{C}^*)^n$ (see [4, Proposition 3.7]), and we call it the *regularized* BC_n*-type Jackson integral*. In particular, if we assume $s = n$ we call $\langle\!\langle 1, \xi \rangle\!\rangle_{\rm G}$ the *regularized Jackson integral of Gustafson's* C_n*-type*, which is, in particular, a constant not depending on $\xi \in (\mathbb{C}^*)^n$.

Remark 4.2. For further results on BC_n-type Jackson integrals, see [2, 4, 3, 5, 13, 14], for instance.

Now we state the recurrence relation for Gustafson's sum $\langle\!\langle 1, \xi \rangle\!\rangle_{\rm G}$.

Proposition 4.1. *Suppose $s = n$ and $x = (x_1, \dots, x_n) \in (\mathbb{C}^*)^n$. The sum $\langle\!\langle 1, x \rangle\!\rangle_{\rm G}$ satisfies*

$$T_{a_j} \langle\!\langle 1, x \rangle\!\rangle_{\rm G} = \langle\!\langle 1, x \rangle\!\rangle_{\rm G} \frac{\prod\limits_{m=1}^{2n+2} (1 - a_j^{-1} a_m^{-1})}{(1 - a_j^{-2})(1 - a_1^{-1} a_2^{-1} \cdots a_{2n+2}^{-1})} \qquad \textit{for} \qquad j = 1, 2, \dots, 2n+2.$$

Proof. We assume $s = n$ for the basis $\{v_0, v_1, \dots, v_{n-1}\}$ of the BC_1-type Jackson integral. Let P be the transition matrix from the basis $\{v_0, v_1, \dots, v_{n-1}\}$ to $\{\chi_{(n-1)}, \chi_{(n-2)}, \dots, \chi_{(0)}\}$:

$$(\chi_{(n-1)}, \chi_{(n-2)}, \dots, \chi_{(0)}) = (v_0, v_1, \dots, v_{n-1}) P,$$

where $\chi_{(i)}$ is the irreducible character of type C_1 defined by

$$\chi_{(i)}(z) = \frac{z^{i+1} - z^{-i-1}}{z - z^{-1}} \qquad \text{for} \qquad i = 0, 1, 2, \dots. \tag{4.8}$$

From (4.6) it follows that

$$T_{a_j}(\langle\!\langle \chi_{(n-1)}, \xi \rangle\!\rangle, \dots, \langle\!\langle \chi_{(0)}, \xi \rangle\!\rangle) = (\langle\!\langle \chi_{(n-1)}, \xi \rangle\!\rangle, \dots, \langle\!\langle \chi_{(0)}, \xi \rangle\!\rangle) P^{-1} \bar{B} P, \tag{4.9}$$

so that

$$T_{a_j} \det \big(\langle\!\langle \chi_{(n-i)}, x_j \rangle\!\rangle\big)_{1 \le i,j \le n} = \det \big(\langle\!\langle \chi_{(n-i)}, x_j \rangle\!\rangle\big)_{1 \le i,j \le n} \det \bar{B} \tag{4.10}$$

for $x = (x_1, \dots, x_n) \in (\mathbb{C}^*)^n$. By definition, the relation between the determinant of the BC_1-type Jackson integrals and the Jackson integral of Gustafson's C_n-type itself is given as

$$\det \big(\langle\!\langle \chi_{(n-i)}, x_j \rangle\!\rangle\big)_{1 \le i,j \le n} = \langle\!\langle 1, x \rangle\!\rangle_{\rm G} \prod_{1 \le j < k \le n} \frac{\theta(x_j/x_k)\theta(x_j x_k)}{x_j}, \tag{4.11}$$

which is also referred to in [14]. From (4.10) and (4.11), we obtain $T_{a_j} \langle\!\langle 1, x \rangle\!\rangle_{\rm G} = \langle\!\langle 1, x \rangle\!\rangle_{\rm G} \det \bar{B}$, where $\det \bar{B}$ has already been given in (4.7). ■

Remark 4.3. The explicit form of the coefficient matrix of the system (4.9) is given in [1] or [3].

Corollary 4.2 (Gustafson [10]). *Suppose $s=n$ and $x=(x_1,\dots,x_n)\in(\mathbb{C}^*)^n$. Then the sum $\langle\!\langle 1,x\rangle\!\rangle_{\mathrm{G}}$ is written as*

$$\langle\!\langle 1,x\rangle\!\rangle_{\mathrm{G}}=(1-q)^n\frac{(q)_\infty^n\prod\limits_{1\le i<j\le 2n+2}\big(qa_i^{-1}a_j^{-1}\big)_\infty}{\big(qa_1^{-1}a_2^{-1}\cdots a_{2n+2}^{-1}\big)_\infty}.$$

Proof. By repeated use of the recurrence relation in Proposition 4.1, using the asymptotic behavior of the Jackson integral as the boundary condition of the recurrence relation, we eventually obtain Corollary 4.2. See [12] for further details about the proof. ■

Remark 4.4. From (4.11) and Corollary 4.2, we see

$$\det\big(\langle\!\langle\chi_{(s-i)},x_j\rangle\!\rangle\big)_{1\le i,j\le s}=(1-q)^s\frac{(q)_\infty^s\prod\limits_{1\le i<j\le 2s+2}\big(qa_i^{-1}a_j^{-1}\big)_\infty}{\big(qa_1^{-1}a_2^{-1}\cdots a_{2s+2}^{-1}\big)_\infty}\prod_{1\le j<k\le s}\frac{\theta(x_j/x_k)\theta(x_jx_k)}{x_j},$$

which is non-degenerate under generic condition. This indicates that the set $\{\chi_{(s-1)},\chi_{(s-2)},\dots,\chi_{(0)}\}$ is linearly independent. And we eventually know the rank of the q-difference system with respect to this basis is s, so are the ranks of the systems (4.2) and (4.6).

5 The case $a_1a_2\cdots a_{2s+2}=1$

5.1 Reflection equation

Theorem 5.1. *Suppose $a_1a_2\cdots a_{2s+2}=1$. Let $v_k(z)$, $k=1,2,\dots,s-1$, be the functions defined by (4.1) for the fixed indices $i_1,i_2,\dots,i_{s-1}\in\{1,2,\dots,2s+2\}$. If $j_1,j_2\notin\{i_1,i_2,\dots,i_{s-1}\}$, then*

$$(\langle e_{j_1}v_1,\xi\rangle,\dots,\langle e_{j_1}v_{s-1},\xi\rangle)=(\langle e_{j_2}v_1,\xi\rangle,\dots,\langle e_{j_2}v_{s-1},\xi\rangle)M,$$

where $M=M_{j_2}NM_{j_1}^{-1}$. Here M_j and N are the matrices defined by

$$M_j=\begin{pmatrix}\gamma_{1,j}&&&&\\ \gamma_{2,j}&1&&&\\ \gamma_{3,j}&&1&&\\ \vdots&&&\ddots&\\ \gamma_{s-1,j}&&&&1\end{pmatrix},\qquad N=\begin{pmatrix}1&\sigma_2&\sigma_3&\cdots&\sigma_{s-1}\\ &\tau_2&&&\\ &&\tau_3&&\\ &&&\ddots&\\ &&&&\tau_{s-1}\end{pmatrix},$$

where the entries of the above matrices are given by

$$\sigma_k=\frac{e(a_{j_1};a_{j_2})}{e(a_{i_k};a_{j_2})},\qquad \tau_k=\frac{e(a_{j_1};a_{i_k})}{e(a_{j_2};a_{i_k})},$$

$$\gamma_{k,j}=\frac{a_j^s\big(1-a_j^2\big)}{a_{i_k}^s\big(1-a_{i_k}^2\big)}\prod_{m=1}^{2s+2}\frac{1-a_{i_k}a_m}{1-a_ja_m}\prod_{\substack{1\le\ell\le s-1\\ \ell\ne k}}\frac{e(a_j;a_{i_\ell})}{e(a_{i_k};a_{i_\ell})}.$$

Moreover,

$$\det M=\frac{a_{j_2}^s\big(1-a_{j_2}^2\big)}{a_{j_1}^s\big(1-a_{j_1}^2\big)}\prod_{m=1}^{2s+2}\frac{1-a_{j_1}a_m}{1-a_{j_2}a_m}. \tag{5.1}$$

Proof. First we will prove the following:

$$(\langle e_{j_1}v_1,\xi\rangle,\langle e_{j_1}v_2,\xi\rangle,\dots,\langle e_{j_1}v_{s-1},\xi\rangle)M_{j_1}=(\langle v_0,\xi\rangle,\langle e_{j_2}v_2,\xi\rangle,\dots,\langle e_{j_2}v_{s-1},\xi\rangle)N, \tag{5.2}$$
$$(\langle e_{j_2}v_1,\xi\rangle,\langle e_{j_2}v_2,\xi\rangle,\dots,\langle e_{j_2}v_{s-1},\xi\rangle)M_{j_2}=(\langle v_0,\xi\rangle,\langle e_{j_2}v_2,\xi\rangle,\dots,\langle e_{j_2}v_{s-1},\xi\rangle), \tag{5.3}$$

which are equivalent to

$$\sum_{k=1}^{s-1}\gamma_{k,j}\langle e_jv_k,\xi\rangle=\langle v_0,\xi\rangle \tag{5.4}$$

and

$$\langle e_{j_1}v_k,\xi\rangle=\sigma_k\langle v_0,\xi\rangle+\tau_k\langle e_{j_2}v_k,\xi\rangle,\qquad k=2,\dots,s-1. \tag{5.5}$$

Under the condition $a_1a_2\cdots a_{2s+2}=1$, Equation (5.4) is a direct consequence of Theorem 3.1. Equation (5.5) is trivial from the equation

$$e(z;a_i)=e(z;a_j)\frac{e(a_i;a_k)}{e(a_j;a_k)}+e(z;a_k)\frac{e(a_i;a_j)}{e(a_k;a_j)},$$

which was given in (3.3). From (5.2) and (5.3), it follows $M=M_{j_2}NM_{j_1}^{-1}$. Moreover, we obtain

$$\det M=\frac{\det M_{j_1}\det N}{\det M_{j_2}}=\frac{\gamma_{1,j_1}\tau_2\cdots\tau_{s-1}}{\gamma_{1,j_2}}=\frac{a_{j_2}^s\left(1-a_{j_2}^2\right)}{a_{j_1}^s\left(1-a_{j_1}^2\right)}\prod_{m=1}^{2s+2}\frac{1-a_{j_1}a_m}{1-a_{j_2}a_m},$$

which completes the proof. ■

Corollary 5.1. *Suppose $s=2$ and the condition $a_6=\frac{q}{a_1a_2a_3a_4a_5}$. The recurrence relation for the BC_1-type Jackson integral $\langle 1,\xi\rangle$ is*

$$T_{a_j}\langle 1,\xi\rangle=\langle 1,\xi\rangle\frac{q}{a_ja_6}\prod_{\substack{1\le\ell\le 5\\ \ell\ne j}}\frac{1-a_ja_\ell}{1-qa_6^{-1}a_\ell^{-1}}\qquad \textit{for}\qquad j=1,2,\dots,5.$$

Proof. Without loss of generality, it suffices to show that

$$T_{a_1}\langle 1,\xi\rangle=\langle 1,\xi\rangle\frac{q}{a_1a_6}\prod_{\ell=2}^{5}\frac{1-a_1a_\ell}{1-qa_6^{-1}a_\ell^{-1}}. \tag{5.6}$$

Set $J(a_1,a_2,a_3,a_4,a_5,a_6;\xi):=\langle 1,\xi\rangle$. Under the condition $a_1a_2a_3a_4a_5a_6=1$, we have

$$J(qa_1,a_2,a_3,a_4,a_5,a_6;\xi)=J(a_1,a_2,a_3,a_4,a_5,qa_6;\xi)\frac{a_6^2}{a_1^2}\prod_{\ell=2}^{5}\frac{1-a_1a_\ell}{1-a_6a_\ell}$$

from Theorem 5.1 by setting $j_1=1$ and $j_2=6$. We now replace a_6 by $q^{-1}a_6$ in the above equation. Then, under the condition $a_1a_2a_3a_4a_5(q^{-1}a_6)=1$, we have

$$J\left(qa_1,a_2,a_3,a_4,a_5,q^{-1}a_6;\xi\right)=J(a_1,a_2,a_3,a_4,a_5,a_6;\xi)\frac{q}{a_1a_6}\prod_{\ell=2}^{5}\frac{1-a_1a_\ell}{1-qa_6^{-1}a_\ell^{-1}}.$$

Since $T_{a_1}\langle 1,\xi\rangle=J(qa_1,a_2,a_3,a_4,a_5,q^{-1}a_6;\xi)$ under this condition $a_6=q(a_1a_2a_3a_4a_5)^{-1}$, we obtain (5.6), which completes the proof. ■

Corollary 5.2. *Suppose $s = n+1$ and the condition $a_{2n+4} = \frac{q}{a_1a_2\cdots a_{2n+3}}$. Then the recurrence relation for Gustafson's sum $\langle 1, x\rangle_{\mathrm{G}}$ where $x = (x_1, \ldots, x_n) \in (\mathbb{C}^*)^n$ is given by*

$$T_{a_j}\langle 1, x\rangle_{\mathrm{G}} = \langle 1, x\rangle_{\mathrm{G}} \frac{q}{a_ja_{2n+4}} \prod_{\substack{1\le \ell\le 2n+3\\ \ell\neq j}} \frac{1-a_ja_\ell}{1-qa_\ell^{-1}a_{2n+4}^{-1}} \qquad \textit{for} \qquad j = 1, 2, \ldots, 2n+3.$$

Proof. Fix $s = n+1$. For the BC_1-type Jackson integral, we first set

$$J(a_1, a_2, \ldots, a_{2n+4}; x) := \det\big(\langle \chi_{(n-i)}, x_j\rangle\big)_{1\le i,j\le n},$$

where $\chi_{(i)}$ is defined in (4.8), under no condition on $a_1, a_2, \ldots, a_{2n+4}$. By the definition of Φ, we have

$$J(qa_1, a_2, \ldots, a_{2n+4}; x) = \det\big(\langle e_1\chi_{(n-i)}, x_j\rangle\big)_{1\le i,j\le n}.$$

Let Q be the transition matrix from the basis $\{v_1, v_2, \ldots, v_n\}$ to $\{\chi_{(n-1)}, \chi_{(n-2)}, \ldots, \chi_{(0)}\}$, i.e.,

$$(\chi_{(n-1)}, \chi_{(n-2)}, \ldots, \chi_{(0)}) = (v_1, v_2, \ldots, v_n)Q.$$

Under the condition $a_1a_2\cdots a_{2n+4} = 1$, from Theorem 5.1 with $j_1 = 1$ and $j_2 = 2n+4$, it follows that

$$(\langle e_1v_1, \xi\rangle, \ldots, \langle e_1v_n, \xi\rangle) = (\langle e_{2n+4}v_1, \xi\rangle, \ldots, \langle e_{2n+4}v_n, \xi\rangle)M,$$

so that

$$(\langle e_1\chi_{(n-1)}, \xi\rangle, \ldots, \langle e_1\chi_{(0)}, \xi\rangle) = (\langle e_{2n+4}\chi_{(n-1)}, \xi\rangle, \ldots, \langle e_{2n+4}\chi_{(0)}, \xi\rangle)Q^{-1}MQ.$$

This indicates that

$$\det\big(\langle e_1\chi_{(n-i)}, x_j\rangle\big)_{1\le i,j\le n} = \det\big(\langle e_{2n+4}\chi_{(n-i)}, x_j\rangle\big)_{1\le i,j\le n}\det M.$$

From (5.1) and the above equation we have

$$J(qa_1, a_2, \ldots, a_{2n+4}; x) = J(a_1, a_2, \ldots, qa_{2n+4}; x)\left(\frac{a_{2n+4}}{a_1}\right)^{n+1}\prod_{\ell=2}^{2n+3}\frac{1-a_\ell a_1}{1-a_\ell a_{2n+4}},$$

under the condition $a_1a_2\cdots a_{2n+4} = 1$. We now replace a_{2n+4} by $q^{-1}a_{2n+4}$ in the above equation. Then we have

$$J\left(qa_1, a_2, \ldots, q^{-1}a_{2n+4}; x\right) = J(a_1, a_2, \ldots, a_{2n+4}; x)\frac{q}{a_1a_{2n+4}}\prod_{\ell=2}^{2n+3}\frac{1-a_\ell a_1}{1-qa_\ell^{-1}a_{2n+4}^{-1}},$$

under the condition $a_1a_2\cdots(q^{-1}a_{2n+4}) = 1$. Since

$$T_{a_1}\det\big(\langle \chi_{(n-i)}, x_j\rangle\big)_{1\le i,j\le n} = J\left(qa_1, a_2, \ldots, q^{-1}a_{2n+4}; x\right)$$

if $a_1a_2\cdots a_{2n+4} = q$, we have

$$T_{a_1}\det\big(\langle \chi_{(n-i)}, x_j\rangle\big)_{1\le i,j\le n} = \det\big(\langle \chi_{(n-i)}, x_j\rangle\big)_{1\le i,j\le n}\frac{q}{a_1a_{2n+4}}\prod_{\ell=2}^{2n+3}\frac{1-a_\ell a_1}{1-qa_\ell^{-1}a_{2n+4}^{-1}}.$$

On the other hand, if $x=(x_1,\dots,x_n)\in(\mathbb{C}^*)^n$, then, by definition we have

$$\det\big(\langle\chi_{(n-i)},x_j\rangle\big)_{1\le i,j\le n}=\langle 1,x\rangle_{\mathrm{G}},$$

which is also referred to in [14]. Therefore, under the condition $a_{2n+4}=q(a_1a_2\cdots a_{2n+3})^{-1}$ we obtain

$$T_{a_1}\langle 1,x\rangle_{\mathrm{G}}=\langle 1,x\rangle_{\mathrm{G}}\frac{q}{a_1a_{2n+4}}\prod_{\ell=2}^{2n+3}\frac{1-a_\ell a_1}{1-qa_\ell^{-1}a_{2n+4}^{-1}}.$$

Since the same argument holds for parameters $a_2,\dots,a_{2n+3}$, we can conclude Corollary 5.2. ■

Remark 5.1. If we take $\xi=a_i$, $i=1,\dots,6$, and add the terminating condition $a_1a_2=q^{-N}$, $N=1,2,\dots$, to the assumptions of Corollary 5.1, then the finite product expression of $\langle 1,\xi\rangle$, which is equivalent to Jackson's formula for terminating ${}_8\phi_7$ series [8, p. 43, equation (2.6.2)], is obtained from finite repeated use of Corollary 5.1. In the same way, if we take a suitable x and add the terminating condition to the assumptions of Corollary 5.2, then the finite product expression of $\langle 1,x\rangle_{\mathrm{G}}$, which is equivalent to the Jackson type formula for terminating multiple ${}_8\phi_7$ series (see [7, Theorem 4] or [6, p. 231, equation (4.4)], for instance), is obtained from finite repeated use of Corollary 5.2.

5.2 Application

The aim of this subsection is to give a simple proof of the following propositions proved by Nassrallah and Rahman [16] and Gustafson [9].

Proposition 5.1 (Nassrallah–Rahman). *Assume $|a_i|<1$ for $1\le i\le 5$. If $a_6=\frac{q}{a_1a_2a_3a_4a_5}$, then*

$$\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}\frac{\big(qa_6^{-1}z\big)_\infty\big(qa_6^{-1}z^{-1}\big)_\infty\big(z^2\big)_\infty\big(z^{-2}\big)_\infty}{\prod_{i=1}^{5}(a_iz)_\infty\big(a_iz^{-1}\big)_\infty}\frac{dz}{z}=\frac{2\prod_{k=1}^{5}\big(qa_6^{-1}a_k^{-1}\big)_\infty}{(q)_\infty\prod_{1\le i<j\le 5}(a_ia_j)_\infty},\tag{5.7}$$

where $\mathbb{T}$ is the unit circle taken in the positive direction.

Proof. We denote the left-hand side of (5.7) by $I(a_1,a_2,a_3,a_4,a_5)$. By residue calculation,

$$\begin{aligned}I(a_1,a_2,a_3,a_4,a_5)&=\sum_{k=1}^{5}\sum_{\nu=0}^{\infty}\operatorname*{Res}_{z=a_kq^\nu}\left[\frac{\theta\big(qa_6^{-1}z^{-1}\big)\theta\big(z^{-2}\big)}{z\prod_{m=1}^{5}\theta\big(a_mz^{-1}\big)}z(1-z^2)\prod_{m=1}^{6}\frac{\big(qa_m^{-1}z\big)_\infty}{(a_mz)_\infty}\right]\frac{dz}{z}\qquad(5.8)\\&=\sum_{k=1}^{5}\left[\operatorname*{Res}_{z=a_k}\frac{\theta\big(qa_6^{-1}z^{-1}\big)\theta\big(z^{-2}\big)}{z\prod_{m=1}^{5}\theta\big(a_mz^{-1}\big)}\frac{dz}{z}\right]\int_0^{a_k\infty}z(1-z^2)\prod_{m=1}^{6}\frac{\big(qa_m^{-1}z\big)_\infty}{(a_mz)_\infty}\frac{d_qz}{z}\\&=\sum_{k=1}^{5}R_k\langle 1,a_k\rangle,\qquad(5.9)\end{aligned}$$

where

$$R_k:=\operatorname*{Res}_{z=a_k}\frac{\theta\big(qa_6^{-1}z^{-1}\big)\theta\big(z^{-2}\big)}{z\prod_{m=1}^{5}\theta\big(a_mz^{-1}\big)}\frac{dz}{z}=\frac{\theta\big(qa_6^{-1}a_k^{-1}\big)\theta\big(a_k^{-2}\big)}{(q)_\infty^2a_k\prod_{1\le m\le 5,\ m\ne k}\theta\big(a_ma_k^{-1}\big)},$$

whose recurrence relation is

$$T_{a_j}R_k = \left(q^{-1}a_ja_6\right)R_k \tag{5.10}$$

for $1 \le j,k \le 5$, which is obtained using (2.2). From (5.9), (5.10) and Corollary 5.1, we obtain the recurrence relation for $I(a_1,a_2,a_3,a_4,a_5)$ as

$$T_{a_j}I(a_1,a_2,a_3,a_4,a_5) = I(a_1,a_2,a_3,a_4,a_5)\prod_{\substack{1\le\ell\le5\\ \ell\ne j}}\frac{1-a_ja_\ell}{1-qa_6^{-1}a_\ell^{-1}}.$$

By repeated use of the above relation, we obtain

$$\begin{aligned}I(a_1,a_2,a_3,a_4,a_5) &= \frac{\prod\limits_{k=1}^{5}\left(qa_6^{-1}a_k^{-1}\right)_{2N}}{\prod\limits_{1\le i<j\le5}(a_ia_j)_{2N}}I\left(q^Na_1,q^Na_2,q^Na_3,q^Na_4,q^Na_5\right)\\ &= \frac{\prod\limits_{k=1}^{5}\left(qa_6^{-1}a_k^{-1}\right)_{\infty}}{\prod\limits_{1\le i<j\le5}(a_ia_j)_{\infty}}\lim_{N\to\infty}I\left(q^Na_1,q^Na_2,q^Na_3,q^Na_4,q^Na_5\right)\end{aligned}$$

and

$$\lim_{N\to\infty}I\left(q^Na_1,q^Na_2,q^Na_3,q^Na_4,q^Na_5\right) = \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}\left(z^2\right)_\infty\left(z^{-2}\right)_\infty\frac{dz}{z} = \frac{2}{(q)_\infty}.$$

This completes the proof. ■

Remark 5.2. Strictly speaking, the residue calculation (5.8) requires that

$$I_\varepsilon := \frac{1}{2\pi\sqrt{-1}}\int_{|z|=\varepsilon}\frac{\left(qa_6^{-1}z\right)_\infty\left(qa_6^{-1}z^{-1}\right)_\infty\left(z^2\right)_\infty\left(z^{-2}\right)_\infty}{\prod\limits_{i=1}^{5}(a_iz)_\infty\left(a_iz^{-1}\right)_\infty}\frac{dz}{z} \to 0 \qquad \text{if} \qquad \varepsilon\to0, \tag{5.11}$$

which can be shown in the following way. We first take $\varepsilon = q^N\varepsilon'$ for $\varepsilon' > 0$ and positive integer N. If we put

$$F(z) := \frac{\left(qa_6^{-1}z\right)_\infty\left(qa_6^{-1}z^{-1}\right)_\infty\left(z^2\right)_\infty\left(z^{-2}\right)_\infty}{\prod\limits_{i=1}^{5}(a_iz)_\infty\left(a_iz^{-1}\right)_\infty},$$

then we have $F(z) = zG_1(z)G_2(z)$, where

$$G_1(z) = \frac{\theta\left(qa_6^{-1}z^{-1}\right)\theta\left(z^{-2}\right)}{z\prod\limits_{i=1}^{5}\theta\left(a_iz^{-1}\right)}, \qquad G_2(z) = \left(1-z^2\right)\prod_{i=1}^{6}\frac{\left(qa_i^{-1}z\right)_\infty}{(a_iz)_\infty}.$$

Since $G_1(z)$ is a continuous function on the compact set $|z| = \varepsilon'$ and is invariant under the q-shift $z\to qz$ under the condition $a_6 = q(a_1a_2a_3a_4a_5)^{-1}$, $|G_1(z)|$ is bounded on $|z| = q^N\varepsilon'$. $|G_2(z)|$ is also bounded because $G_2(z)\to1$ if $z\to0$. Thus there exists $C>0$ such that $|F(z)| < C|z|$. If we put $z = \varepsilon e^{2\pi\sqrt{-1}\tau}$, then

$$|I_\varepsilon| < \int_0^1\left|F\left(\varepsilon e^{2\pi\sqrt{-1}\tau}\right)\right|d\tau < C\int_0^1\left|\varepsilon e^{2\pi\sqrt{-1}\tau}\right|d\tau = C\varepsilon\to0, \qquad \varepsilon\to0,$$

which proves (5.11).

Proposition 5.2 (Gustafson [9]). *Assume $|a_i| < 1$ for $1 \le i \le 2n+3$. If $a_{2n+4} = \frac{q}{a_1 a_2 \cdots a_{2n+3}}$, then*

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{\mathbb{T}^n} \prod_{i=1}^{n} \frac{\left(qa_{2n+4}^{-1}z_i\right)_\infty \left(qa_{2n+4}^{-1}z_i^{-1}\right)_\infty \left(z_i^2\right)_\infty \left(z_i^{-2}\right)_\infty}{\prod\limits_{k=1}^{2n+3} (a_k z_i)_\infty \left(a_k z_i^{-1}\right)_\infty}$$

$$\times \prod_{1\le i<j\le n} (z_i z_j)_\infty \left(z_i z_j^{-1}\right)_\infty \left(z_i^{-1} z_j\right)_\infty \left(z_i^{-1} z_j^{-1}\right)_\infty \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$$

$$= \frac{2^n n! \prod\limits_{k=1}^{2n+3} \left(qa_{2s+4}^{-1} a_k^{-1}\right)_\infty}{(q)_\infty^n \prod\limits_{1\le i<j\le 2s+3} (a_i a_j)_\infty}, \tag{5.12}$$

where $\mathbb{T}^n$ is the n-fold direct product of the unit circle traversed in the positive direction.

The proof below is based on an idea using residue computation due to Gustafson [10], which is done for the case of the hypergeometric integral under no balancing condition. Here we will show that his residue method is still effective even for the integral under the balancing condition $a_1 a_2 \cdots a_{2n+4} = q$. In particular, this is different from his proof in [9].

Proof. Let L be the set of indices defined by

$$L := \{\lambda = (\lambda_1, \dots, \lambda_n);\ 1 \le \lambda_1 < \lambda_2 < \cdots < \lambda_n \le 2n+3\}.$$

Set $a_{(\mu)} := (a_{\mu_1}, \dots, a_{\mu_n}) \in (\mathbb{C}^*)^n$ for $\mu = (\mu_1, \dots, \mu_n) \in L$. We denote the left-hand side of (5.12) by $I(a_1, a_2, \dots, a_{2n+3})$. By residue calculation, we have

$$I(a_1, a_2, \dots, a_{2n+3}) = \sum_{\mu \in L} R_\mu \langle 1, a_{(\mu)} \rangle_{\mathrm{G}}, \tag{5.13}$$

where the coefficients R_μ, $\mu \in L$, are

$$R_\mu := \operatorname*{Res}_{\substack{z_1 = a_{\mu_1} \\ \cdots \\ z_n = a_{\mu_n}}} \left[\prod_{i=1}^{n} \frac{\theta\left(qa_{2n+4}^{-1} z_i^{-1}\right) \theta\left(z_i^{-2}\right)}{z_i \prod\limits_{m=1}^{2n+3} \theta\left(a_m z_i^{-1}\right)} \prod_{1\le j<k\le n} \theta\left(z_j^{-1} z_k\right) \theta\left(z_j^{-1} z_k^{-1}\right) \right] \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}.$$

The recurrence relation for R_μ is

$$T_{a_j} R_\mu = \left(q^{-1} a_j a_{2n+4}\right) R_\mu. \tag{5.14}$$

From (5.13), (5.14) and Corollary 5.2, we obtain the recurrence relation for $I(a_1, a_2, \dots, a_{2n+3})$ as

$$T_{a_j} I(a_1, a_2, \dots, a_{2n+3}) = I(a_1, a_2, \dots, a_{2n+3}) \prod_{\substack{1\le \ell \le 2n+3 \\ \ell \ne j}} \frac{1 - a_j a_\ell}{1 - qa_{2n+4}^{-1} a_\ell^{-1}}.$$

By repeated use of the above relation, we obtain

$$I(a_1, a_2, \dots, a_{2n+3}) = \frac{\prod\limits_{k=1}^{2n+3} \left(qa_{2n+4}^{-1} a_k^{-1}\right)_{2N}}{\prod\limits_{1\le i<j\le 2n+3} (a_i a_j)_{2N}} I\left(q^N a_1, q^N a_2, \dots, q^N a_{2n+3}\right)$$

$$= \frac{\prod_{k=1}^{2n+3} \left(q a_{2n+4}^{-1} a_k^{-1}\right)_\infty}{\prod_{1\le i<j\le 2n+3} (a_i a_j)_\infty} \lim_{N\to\infty} I\left(q^N a_1, q^N a_2, \dots, q^N a_{2n+3}\right)$$

and

$$\lim_{N\to\infty} I\left(q^N a_1, q^N a_2, \dots, q^N a_{2n+3}\right) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{\mathbb{T}^n} \prod_{i=1}^n \left(z_i^2\right)_\infty \left(z_i^{-2}\right)_\infty$$

$$\times \prod_{1\le i<j\le n} (z_i z_j)_\infty \left(z_i z_j^{-1}\right)_\infty \left(z_i^{-1} z_j\right)_\infty \left(z_i^{-1} z_j^{-1}\right)_\infty \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n} = \frac{2^n n!}{(q)_\infty^n}.$$

This completes the proof. ■

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