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The Scattering Problem for a Noncommutative Nonlinear Schrödinger Equation*

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Abstract. We investigate scattering properties of a Moyal deformed version of the nonlinear Schrödinger equation in an even number of space dimensions. With rather weak conditions on the degree of nonlinearity, the Cauchy problem for general initial data has a unique globally defined solution, and also has solitary wave solutions if the interaction potential is suitably chosen. We demonstrate how to set up a scattering framework for equations of this type, including appropriate decay estimates of the free time evolution and the construction of wave operators defined for small scattering data in the general case and for arbitrary scattering data in the rotationally symmetric case.

Key words: noncommutative geometry; nonlinear wave equations; scattering theory; Jacobi polynomials

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1 Introduction

The scattering problem for general nonlinear field equations has been intensively studied for many years with considerable progress, a seminal result being the establishment of asymptotic completeness in energy space for the nonlinear Schrödinger equation in space dimension $n \geq 3$ with interaction $|\varphi|^{p-1}\varphi$, where $1 + 4/n \leq p \leq 1 + 4/(n-2)$, and analogous results for the nonlinear Klein–Gordon equation [8]. Still, many important questions remain unanswered, in particular relating to the existence of wave operators defined on appropriate subspaces of scattering states and to asymptotic completeness for more general interactions, although many partial results have been obtained. We refer to [17, 7] for an overview. One source of difficulties encountered can be traced back to the singular nature of pointwise multiplication of functions with respect to appropriate Lebesgue or Sobolev norms. It therefore appears natural to look for field equations where this part of the problem can be eliminated while preserving as much as possible of the remaining structure. One such possibility is to deform the standard product appropriately, and our purpose in the present article is to exploit this idea.

Deformations of the kind mentioned applied to relativistic field equations have turned out to be of relevance for understanding non-perturbative aspects of string theory, see e.g. [9, 11, 4]. We shall limit ourselves to studying a deformed version of the nonlinear Schrödinger equation in an even number $n = 2d$ of space dimensions. The rather crude Hilbert space techniques [13, 14] we apply allow us to consider polynomial interactions only. We hope that an application of more

*This paper is a contribution to the Special Issue "Noncommutative Spaces and Fields". The full collection is available at http://www.emis.de/journals/SIGMA/noncommutative.html modern techniques [2] will allow a treatment of a larger class of interactions. Our main purpose will be to demonstrate that, for the deformed equation, the Cauchy problem is in a certain sense more regular as compared to the classical equation as well as to set up a natural scattering framework, including appropriate decay estimates of the free time evolution and construction of wave operators under rather mild conditions on the interaction.

The deformed, or noncommutative, version of the nonlinear Schrödinger equation (NCNLS) in $2d$ space dimensions can be written as

$$\big(i\partial_t - \Delta\big)\varphi(x,t) = \varphi \star_\theta F_{\star_\theta}(\varphi^* \star_\theta \varphi)(x,t), \tag{1}$$

where $\star_\theta$ denotes the Moyal product (see below) of functions of $2d$ space variables x, $F_{\star_\theta}$ denotes a real polynomial with respect to this product and $\Delta = -\sum_{i=1}^{2d} \partial_i^2$ is the standard Laplacian in $2d$ variables. The Moyal product considered here is given by

$$f \star_\theta g(x) := (2\pi)^{-2d} \int e^{-iyz} f\left(x - \frac{\Theta}{2}y\right) g(x+z)\, d^{2d}y\, d^{2d}z.$$

The constant skew-symmetric $(2d \times 2d)$-matrix Θ is assumed to be given in the canonical form

$$\Theta = \theta \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix},$$

where I_d denotes the $d \times d$ identity matrix and $\theta > 0$ is called the deformation parameter.

Defining $\varphi_\theta(x,t) = \varphi(\theta^{\frac{1}{2}}x, \theta t)$, we have the scaling identity

$$(\varphi \star_\theta \psi)_\theta = \varphi_\theta \star_1 \psi_\theta.$$

It follows that φ satisfies (1) if and only if

$$\big(i\partial_t - \Delta\big)\varphi_\theta(x,t) = \theta\varphi_\theta \star F_\star(\varphi_\theta^* \star \varphi_\theta)(x,t),$$

where we have dropped the subscript on the $\star$-product when $\theta = 1$.

The $\star$-product is intimately connected to the so-called Weyl quantization map W. This map associates an operator $W(f)$ on $L^2(\mathbb{R}^d)$ to appropriate functions (or distributions) f on $\mathbb{R}^{2d}$ such that the kernel $K_W(f)$ of $W(f)$ is given by

$$K_W(f)(x,y) = (2\pi)^{-d} \int_{\mathbb{R}^d} f\left(\frac{x+y}{2}, p\right) e^{i(x-y)\cdot p}\, dp.$$

It is easily seen that W is an isomorphism from $L^2(\mathbb{R}^{2d})$ onto the Hilbert space $\mathcal{H}$ of Hilbert–Schmidt operators on $L^2(\mathbb{R}^d)$ fulfilling

$$\|W(f)\|_2 = (2\pi)^{-d/2} \|f\|_{L^2(\mathbb{R}^{2d})}, \tag{2}$$

where $\|\cdot\|_2$ denotes the Hilbert–Schmidt norm. Moreover, W maps the space $\mathcal{S}(\mathbb{R}^{2d})$ onto the space of operators whose kernel is a Schwartz function and its relation to the $\star$-product is exhibited by the identity

$$W(f \star g) = W(f)W(g).$$

Further properties of the $\star$-product can be found in e.g. [6].

By use of W, equation (1) can be restated (see also [4]) as a differential equation

$$i\partial_t\phi - 2\sum_{k=1}^{d} [a_k^*, [a_k, \phi]] = \theta\phi F(\phi^*\phi), \tag{3}$$

for the operator-valued function $\phi(t) := W(\varphi_\theta(\cdot, t))$, where we have introduced the creation and annihilation operators

$$a_k = \frac{1}{\sqrt{2}}(x_k + \partial_k) \qquad \text{and} \qquad a_k^* = \frac{1}{\sqrt{2}}(x_k - \partial_k).$$

The operator

$$\boldsymbol{\Delta} = 2\sum_{k=1}^{d} \operatorname{ad} a_k^* \operatorname{ad} a_k,$$

with domain $D(\boldsymbol{\Delta})$, is defined in a natural way as a self-adjoint operator on $\mathcal{H}$ by the relations

$$\boldsymbol{\Delta} = W\Delta W^{-1}, \qquad D(\boldsymbol{\Delta}) = WD(\Delta), \tag{4}$$

where Δ denotes the standard self-adjoint $2d$-dimensional Laplace operator with maximal domain $D(\Delta) = H_2^2(\mathbb{R}^{2d})$.

We shall primarily be interested in globally defined *mild* solutions to the Cauchy problem associated to equation (3), that is continuous solutions $\phi : \mathbb{R} \to \mathcal{H}$ to the corresponding integral equation

$$\phi(t) = e^{-i(t-t_0)\boldsymbol{\Delta}}\phi_0 - i\int_{t_0}^{t} e^{i(s-t)\boldsymbol{\Delta}}\phi(s)F\big(|\phi(s)|^2\big)\, ds. \tag{5}$$

This equation is weaker than (3) in the sense that if $\phi : I \to D(\boldsymbol{\Delta})$ is a continuously differentiable solution to (3) defined on an interval I containing t_0, i.e. a *strong solution* on I, then it also fulfills (5). This latter equation naturally fits into the standard Hilbert space framework for evolution equations, see e.g. [14, 13]. The following theorem is proven in Section 2 below.

Theorem 1. *Let F be an arbitrary polynomial over $\mathbb{R}$.*

a) *For every $\phi_0 \in \mathcal{H}$, the equation* (5) *has a unique continuous solution $\phi : \mathbb{R} \to \mathcal{H}$. For every $\phi_0 \in D(\boldsymbol{\Delta})$, the equation* (3) *has a unique strong solution $\phi : \mathbb{R} \to D(\boldsymbol{\Delta})$ such that $\phi(t_0) = \phi_0$.*

b) *Assume F is a polynomial over $\mathbb{R}$ with positive highest order coefficient that has a unique local minimum x_0 on the positive real line and that $F(x_0) < F(0)$. If θ is large enough, there exist oscillating solutions to* (3) *of the form*

$$\phi(t) = e^{i\omega(t_0 - t)}\phi_0, \qquad t \in \mathbb{R},$$

for suitably chosen initial data $\phi_0 \in D(\boldsymbol{\Delta})$ and frequency $\omega \in \mathbb{R}$.

This result follows by a slight adaptation of the methods of [4, 3]. The essential new aspect of *a*), as compared to the corresponding result for the classical case [17, Theorem 3.2], is its validity for interaction polynomials without restrictions on the degree of nonlinearity. Solitary waves as in b) are well known to exist for the classical nonlinear Schrödinger equation with attractive interaction $-|\varphi|^{p-1}\varphi$ in the range $1 < p < 1 + 4/(n-2)$ for $n \geq 3$ [1, 10, 15].

In order to formulate our main results on the scattering problem we need to introduce appropriate Hilbert spaces and auxiliary norms. By $|n\rangle$, $n = (n_1, \ldots, n_k) \in \mathbb{N}_0^d$ we shall denote the standard orthonormal basis for $L^2(\mathbb{R}^d)$ consisting of eigenstates for the d-dimensional harmonic oscillator, and by (ϕ_{mn}) we denote the matrix representing a bounded operator ϕ with respect to this basis, that is

$$\phi = \sum_{m,n} \phi_{mn}|n\rangle\langle m|, \qquad \phi_{mn} := \langle n|\phi|m\rangle.$$

Let

$$b_{mn} := 1 + |m - n|, \qquad m, n \in \mathbb{N}_0^d,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. For exponents $p \geq 1$, $\alpha \in \mathbb{R}$ and ϕ as above, we define the norms $\|\cdot\|_{p,\alpha}$ by

$$\|\phi\|_{p,\alpha}^p := \sum_{m,n} b_{mn}^{\alpha p} |\phi_{mn}|^p, \quad p < \infty, \qquad \text{and} \qquad \|\phi\|_{\infty,\alpha} := \sup_{m,n}\{b_{mn}^{\alpha}|\phi_{mn}|\}, \tag{6}$$

and we let $\mathcal{L}_{p,\alpha}$ denote the space of operators on $L^2(\mathbb{R}^d)$ for which $\|\cdot\|_{p,\alpha}$ is finite. We note that $\mathcal{H}_\alpha := \mathcal{L}_{2,\alpha}$ is a Hilbert space with scalar product

$$\langle \phi, \psi \rangle_\alpha := \sum_{m,n} b_{mn}^{2\alpha} \overline{\phi_{mn}} \psi_{mn},$$

and, in particular, $\mathcal{H}_0$ equals the space $\mathcal{H}$ of Hilbert–Schmidt operators on $L^2(\mathbb{R}^d)$ with $\|\cdot\|_{2,0} = \|\cdot\|_2$. Clearly, the operator $U_\alpha : \mathcal{H}_\alpha \to \mathcal{H}$ defined by

$$(U_\alpha \phi)_{mn} = b_{mn}^{\alpha} \phi_{mn}, \qquad \phi \in \mathcal{H}_\alpha, \tag{7}$$

is unitary, and the operator $\boldsymbol{\Delta}_\alpha := U_\alpha^* \, \boldsymbol{\Delta} \, U_\alpha$, with domain $D(\boldsymbol{\Delta}_\alpha) := U_\alpha^* D(\boldsymbol{\Delta})$, is self-adjoint on $\mathcal{H}_\alpha$.

We use the notation $\|\cdot\|_{\rm op}$ for the standard operator norm and define the norm $\|\cdot\|_a$ by

$$\|\phi\|_a := \|\widetilde{\phi}\|_{\rm op},$$

whenever the operator

$$\widetilde{\phi} := \sum_{m,n} |\phi_{mn}| \, |n\rangle\langle m|$$

is bounded.

With this notations we have the following decay estimate for the free propagation:

Theorem 2. *For $\alpha \geq 0$ and $\beta > d$ there exists a constant $c_\beta > 0$ such that*

$$\| e^{-it\boldsymbol{\Delta}_\alpha} \phi \|_a \leq c_\beta (1 + |t|)^{-\frac{d}{2}} \|\phi\|_{1,\beta}, \qquad \phi \in \mathcal{H}_\alpha \cap \mathcal{L}_{1,\beta}. \tag{8}$$

This estimate should be compared to the standard one used for the classical nonlinear wave equations with the L^∞-norm on the left and a L^1-Sobolev norm on the right, applied to functions of $2d$ space variables. In this case, one obtains rather trivially a decay exponent d instead of $d/2$. However, those norms are not well behaved with respect to the $\star$-product and cannot be used for our purposes. Instead, we have to work a bit harder to establish (8) using uniform estimates on the classical Jacobi polynomials. This is accomplished in Section 4.

Our last main result concerns the existence of wave operators defined on a scattering subspace $\Sigma_\alpha \subset \mathcal{H}_\alpha$. In order to define the latter we introduce, for fixed $\alpha \geq 0$, the scattering norm of $\phi \in \mathcal{H}_\alpha$ by

$$|||\phi|||_\alpha := \|\phi\|_{2,\alpha} + \sup_{t \in \mathbb{R}} |t|^{\frac{d}{2}} \|e^{-it\boldsymbol{\Delta}_\alpha} \phi\|_a$$

and set

$$\Sigma_\alpha := \{\phi \in \mathcal{H}_\alpha \mid |||\phi|||_\alpha < \infty\} \ .$$

From (6), we immediately see that $\mathcal{L}_{p,\alpha} \subset \mathcal{L}_{q,\beta}$, if $p \le q$ and $\alpha p \ge \beta q$, so that Theorem 2 gives, in particular, the inclusion $\mathcal{L}_{1,\beta} \subset \Sigma_\alpha$ if $\beta > \max\{d, 2\alpha\}$.

The relevant integral equations to solve in a scattering situation correspond to initial/final data $\phi_\pm$ at $t = \pm\infty$ and take the form

$$\phi(t) = e^{-it\boldsymbol{\Delta}_\alpha}\phi_- - i\int_{-\infty}^{t} e^{i(s-t)\boldsymbol{\Delta}_\alpha}\phi(s)F\big(|\phi(s)|^2\big)\,ds, \tag{9}$$

and

$$\phi(t) = e^{-it\boldsymbol{\Delta}_\alpha}\phi_+ + i\int_{t}^{+\infty} e^{i(s-t)\boldsymbol{\Delta}_\alpha}\phi(s)F\big(|\phi(s)|^2\big)\,ds, \tag{10}$$

respectively, where F has been redefined to include θ. *We assume here for simplicity that the polynomial F has no constant term*, since such a term could trivially be incorporated by subtracting it from $\boldsymbol{\Delta}_\alpha$ in the preceding formulas. The first problem then is to determine spaces of initial/final data $\phi_\pm$, such that the equations above have a unique global solution and such that these solutions behave as free solutions for $t \to \pm\infty$. To this end we establish the following in Section 5.

Theorem 3. *Let $\alpha > 2d$ and assume that the polynomial F has no constant term and, in addition, no linear term if $d = 1$ or $d = 2$.*

Then there exists $\delta > 0$ such that for every $\phi_\pm \in \Sigma_\alpha$ with $|||\phi_\pm|||_\alpha < \delta$, the equations (9) *and* (10) *have unique globally defined continuous solutions $\phi^\pm : \mathbb{R} \to \Sigma_\alpha$ fulfilling*

$$\|\phi^\pm(t) - e^{-it\boldsymbol{\Delta}_\alpha}\phi_\pm\|_{2,\alpha} \to 0, \qquad \textit{for} \;\; t \to \pm\infty. \tag{11}$$

If, furthermore, F has no linear term, then we have

$$|||e^{it\boldsymbol{\Delta}_\alpha}\phi^\pm(t) - \phi_\pm|||_\alpha \to 0, \qquad \textit{for} \;\; t \to \pm\infty. \tag{12}$$

Remark 1. We do not know at present whether the assumption that F be without linear term is strictly necessary or is merely an artifact due to the crudeness of the methods applied. A similar, but weaker, limitation on the behaviour of F close to 0 appears in the corresponding result for the classical NLS quoted in [17, Teorem 6.6], where the specification of the scattering spaces is, however, not made explicit.

This result allows the definition of injective *wave operators* $\Omega_\pm : \{\phi_\pm \in \Sigma_\alpha|\; |||\phi_\pm|||_\alpha \le \delta\} \to \Sigma_\alpha$ for small data at $\pm\infty$ in the standard fashion by

$$\Omega_\pm\phi_\pm = \phi^\pm(0). \tag{13}$$

It even allows a definition of a *scattering operator* $S = \Omega_+^{-1}\Omega_-$ for sufficiently small data at $-\infty$, see Remark 3 below. Existence of wave operators defined for arbitrary data in $\Sigma_\pm$ would follow if the corresponding Cauchy problem

$$\phi(t) = e^{-i(t-t_0)\boldsymbol{\Delta}_\alpha}\phi_0 - i\int_{t_0}^{t} e^{i(s-t)t\boldsymbol{\Delta}_\alpha}\phi(s)F\big(|\phi(s)|^2\big)\,ds,$$

has global solutions for all $\psi_0 \in \mathcal{H}_\alpha$. The proof of the global existence result of Theorem 1 relies on the conservation of $\|\cdot\|_2$-norm, which does not hold for the $\|\cdot\|_{2,\alpha}$-norm if $\alpha \ne 0$. Hence, we do not at present know how to treat large scattering data except for the case where ϕ is assumed to be a diagonal operator w.r.t. the harmonic oscillator basis $\{|n\rangle\}$, and in particular for the rotationally symmetric case. The results for this case are reported in Section 6.

2 Existence of global solutions

In this section we give a proof of Theorem 1.

Proof of part *a*). This follows by a straight-forward application of well known techniques, see e.g. [13]. Hence, we only indicate the main line of argument.

Iterating the inequality

$$\|\phi_1\phi_2 - \psi_1\psi_2\|_2 \leq \|\phi_1 - \psi_1\|_2\|\phi_2\|_2 + \|\psi_1\|_2\|\phi_2 - \psi_2\|_2, \qquad \phi_1, \phi_2, \psi_1, \psi_2 \in \mathcal{H},$$

and using $\|\phi\|_2 = \|\phi^*\|_2$ we obtain

$$\|\phi F(\phi^*\phi) - \psi F(\psi^*\psi)\|_2 \leq C_1(\|\phi\|_2, \|\psi\|_2)\|\phi - \psi\|_2, \tag{14}$$

where C_1 is a polynomial with positive coefficients, in particular an increasing function of $\|\phi\|_2$ and $\|\psi\|_2$. By Corollary 1 to Theorem 1 of [13], this suffices to ensure existence of a local continuous solution $\phi :]t_0 - T, t_0 + T[\to \mathcal{H}$ to (5), where T is a decreasing function of $\|\phi_0\|_2$.

Using that the mappings $\phi \to [a_k, \phi]$ and $\phi \to [a_k^*, \phi]$ are derivations on $W^{-1}\mathcal{S}(\mathbb{R}^{2d})$, it follows that $\mathbf{\Delta}(\phi_1 \cdots \phi_n)$ is a polynomial in ϕ_i, $[a_k, \phi_i]$, $[a_k^*, \phi_i]$, $\mathbf{\Delta}\phi_i$, $i = 1, \ldots, n$, $k = 1, \ldots, d$. Since

$$\|[a_k, \phi]\|_2^2 = \mathrm{Tr}(\phi^*[a_k^*, [a_k, \phi]]) = \mathrm{Tr}(\phi^*\mathbf{\Delta}\phi) \leq \|\phi\|_2\|\mathbf{\Delta}\phi\|_2, \tag{15}$$

we conclude as above that

$$\|\mathbf{\Delta}(\phi F(\phi^*\phi) - \psi F(\psi^*\psi))\|_2 \leq C_2(\|\phi\|_2, \|\psi\|_2, \|\mathbf{\Delta}\phi\|_2, \|\mathbf{\Delta}\psi\|_2)(\|\phi - \psi\|_2 + \|\mathbf{\Delta}(\phi - \psi)\|_2),$$

where C_2 is an increasing function of its arguments. In the first place, this inequality holds for $\phi, \psi \in W^{-1}\mathcal{S}(\mathbb{R}^{2d})$, but since $\mathbf{\Delta}$ equals the closure of its restriction to $W^{-1}\mathcal{S}(\mathbb{R}^{2d})$, it holds for all ϕ, ψ in its domain $D(\mathbf{\Delta})$. By Theorem 1 in [13] (or rather its proof) it follows that for each $\phi_0 \in D(\mathbf{\Delta})$ there exists a unique strong solution $\phi :]t_0 - T, t_0 + T[\to D(\mathbf{\Delta})$ to equation (3) with $\phi(t_0) = \phi_0$, where $T > 0$ can be chosen as a decreasing function of $\|\phi_0\|_2$ and $\|\mathbf{\Delta}\phi_0\|_2$.

For strong solutions the 2-norm is conserved:

$$\frac{d}{dt}\|\phi(t)\|_2^2 = -i\mathrm{Tr}\big[(\mathbf{\Delta}\phi - \phi F(\phi^*\phi))^*\phi\big] + i\mathrm{Tr}\big[\phi^*(\mathbf{\Delta}\phi - \phi F(\phi^*\phi))\big] = 0. \tag{16}$$

Furthermore, since the polynomial $\mathbf{\Delta}(\phi F(\phi^*\phi))$ considered above, is a sum of monomials each of which either contains one factor $\mathbf{\Delta}\phi$ or $\mathbf{\Delta}\phi^*$ or two factors of the form $[a_k, \phi]$, $[a_k^*, \phi]$, $[a_k, \phi^*]$, $[a_k^*, \phi^*]$, we conclude from (15) that

$$\|\mathbf{\Delta}(\phi F(\phi^*\phi))\|_2 \leq C_3(\|\phi\|_2)\|\mathbf{\Delta}\phi\|_2, \qquad \phi \in D(\mathbf{\Delta}), \tag{17}$$

where C_3 is an increasing function of its argument.

Using (16) and (17) it follows from Theorem 2 of [13] that strong solutions are globally defined, i.e. we can choose $T = \infty$. Finally, using (14) and (16) again one shows via Corollary 2 to Theorem 14 of [13] that the weak solutions are likewise globally defined. This proves *a*). ■

Proof of part *b*). Let G be the polynomial vanishing at $x = 0$ and satisfying $G' = F$. Under the stated assumptions on F it follows that the polynomial $G(x^2) - F(x_0)x^2$ is positive except at $x = 0$, which is a second order zero, and its derivative is positive on $\mathbb{R}_+$ except for a zero at $x = \sqrt{x_0}$. Hence, for $\epsilon > 0$ sufficiently small, the polynomial

$$V(x) = \frac{1}{2}\big(G\big(x^2\big) - (F(x_0) + \epsilon)x^2\big),$$

is positive, except for a second order zero at $x = 0$, and has a a single local minimum on $\mathbb{R}_+$. Thus $V(x)$ fulfills the assumptions of Theorem 1 of [4] implying the existence of a self-adjoint solution $\phi_0 \in D(\boldsymbol{\Delta})$ to the equation

$$\boldsymbol{\Delta}\phi + \theta V'(\phi) = 0,$$

if θ is sufficiently large. It then follows that $\phi(t) = e^{i\omega(t_0-t)}\phi_0$ is a strong solution of (3), with $\omega = \theta(F(x_0) + \epsilon)$. This proves *b*). ■

3 Some norm inequalities

In preparation for the proofs of Theorems 2 and 3 we collect in this section a few useful lemmas.

Lemma 1. *For $\alpha < -d$, the matrix $\{b^\alpha_{mn}\}$ represents a bounded operator B_α on $L^2(\mathbb{R}^d)$ with respect to the basis $\{|n\rangle\}$.*

Proof. For any $x, y \in L^2(\mathbb{R}^d)$, consider their coordinate sequences $\{x_n\}, \{y_n\} \in \ell^2(\mathbb{N}_0^d)$ w.r.t. the basis $\{|n\rangle\}$. We have

$$\begin{aligned}|\langle x, B_\alpha y\rangle| = \left|\sum_{m,n} \overline{x_m} b^\alpha_{mn} y_n\right| &\le \sum_{m,n}(1 + |m-n|)^\alpha |x_m||y_n| \\ &\le \sum_{m,k}(1 + |k|)^\alpha |x_m||y_{k+m}| \le \sum_k (1+|k|)^\alpha \|x\|_{L^2}\|y\|_{L^2},\end{aligned}$$

which concludes the proof. ■

Lemma 2. *For any $\alpha > d$, there exists a constant $C_\alpha > 0$ such that the following holds:*

$$\|\phi\|_a \le C_\alpha \|\phi\|_{\infty,\alpha}, \qquad \phi \in \mathcal{L}_{\infty,\alpha}.$$

Proof. With notation as in the previous proof we have, for $\phi \in \mathcal{L}_{\infty,\alpha}$,

$$\left|\sum_{m,n} |\phi_{mn}| \overline{x_n}\, y_m\right| \le \sum_{m,n} \left|b^\alpha_{mn}\phi_{mn}\right|\left|b^{-\alpha}_{mn}\overline{x_n} y_m\right| \le \sup_{m,n}\left|b^\alpha_{mn}\phi_{mn}\right| \|B_{-\alpha}\|_{\mathrm{op}} \|x\|_{L^2}\|y\|_{L^2},$$

which evidently implies the claim by Lemma 1. ■

Lemma 3. *For any $\alpha < -d$ there exists a constant $C'_\alpha > 0$ such that*

$$\|\phi\|_{1,\alpha} \le C'_\alpha \|\phi\|_1, \qquad \phi \in \mathcal{L}_1,$$

where $\mathcal{L}_1$ denotes the space of trace class operators and $\|\cdot\|_1$ is the standard trace-norm.

Proof. Writing the trace class operator ϕ as $\frac{1}{2}(\phi + \phi^*) + \frac{i}{2}(i\phi^* - i\phi)$ we may assume that ϕ is self-adjoint. Then, writing $\phi = \phi_+ - \phi_-$ where $\phi_\pm$ are positive operators each of which has trace norm at most that of ϕ, we can assume ϕ is positive. In this case the Cauchy–Schwarz inequality gives

$$|\phi_{kl}| \le \left(\phi_{kk}\phi_{ll}\right)^{1/2},$$

and hence

$$\sum_{k,l} b^\alpha_{kl}|\phi_{kl}| \le \sum_{k,l} b^\alpha_{kl}\phi^{1/2}_{kk}\phi^{1/2}_{ll} \le \|B_\alpha\|_{\mathrm{op}} \sum_k \phi_{kk} = \|B_\alpha\|_{\mathrm{op}}\|\phi\|_1,$$

which proves the claim thanks to Lemma 1. ■

Lemma 4. *a) For any $\alpha \geq 0$ there exists a constant $C_{1,\alpha} > 0$ such that*

$$\|\phi\psi\|_{2,\alpha} \leq C_{1,\alpha}(\|\phi\|_{2,\alpha}\|\psi\|_a + \|\phi\|_a\|\psi\|_{2,\alpha}), \qquad \phi, \psi \in \mathcal{L}_{2,\alpha}. \tag{18}$$

b) For any $\alpha > d$ there exists a constant $C_{2,\alpha} > 0$ such that

$$\|\phi\psi\|_{1,\alpha} \leq C_{2,\alpha}(\|\phi\|_{2,2\alpha}\|\psi\|_2 + \|\phi\|_2\|\psi\|_{2,2\alpha}), \qquad \phi, \psi \in \mathcal{L}_{2,2\alpha}. \tag{19}$$

Proof. First, note that for $\alpha \geq 0$

$$b^\alpha_{mn} \leq c(\alpha)\,(b^\alpha_{mk} + b^\alpha_{kn}), \qquad m, n, k \in \mathbb{N}_0^d, \tag{20}$$

where the constant $c(\alpha)$ depends only on α. We then have

$$\begin{aligned}
\|\phi\psi\|^2_{2,\alpha} &= \sum_{m,n} b^{2\alpha}_{mn}\Big|\sum_k \phi_{mk}\psi_{kn}\Big|^2 \leq C\sum_{m,n}\Big(\sum_k b^\alpha_{mk}|\phi_{mk}|\,|\psi_{kn}| + \sum_k b^\alpha_{kn}|\phi_{mk}|\,|\psi_{kn}|\Big)^2\\
&\leq 2C\sum_{m,n}\Big(\sum_k b^\alpha_{mk}|\phi_{mk}|\,|\psi_{kn}|\Big)^2 + 2C\sum_{m,n}\Big(\sum_k b^\alpha_{kn}|\phi_{mk}|\,|\psi_{kn}|\Big)^2\\
&= 2C\left(\|(U_\alpha\widetilde{\phi})\widetilde{\psi}\|^2_2 + \|\widetilde{\phi}(U_\alpha\widetilde{\psi})\|^2_2\right)\\
&\leq 2C\left(\|U_\alpha\widetilde{\phi}\|^2_2\|\widetilde{\psi}\|^2_{\mathrm{op}} + \|U_\alpha\widetilde{\psi}\|^2_2\|\widetilde{\phi}\|^2_{\mathrm{op}}\right) \leq 2C\left(\|\phi\|_{2,\alpha}\|\psi\|_a + \|\psi\|_{2,\alpha}\|\phi\|_a\right)^2,
\end{aligned}$$

where $C = c(\alpha)^2$ and $U_\alpha : \mathcal{H}_\alpha \to \mathcal{H}_0$ is the unitary operator defined by (7). This establishes (18).

Using again (20), we have

$$\begin{aligned}
\|\phi\psi\|_{1,\alpha} &= \sum_{m,n} b^\alpha_{mn}\Big|\sum_k \phi_{mk}\psi_{kn}\Big| \leq c(2\alpha)\sum_{m,n,k} b^{-\alpha}_{mn}\left(|b^{2\alpha}_{mk}\phi_{mk}|\,|\psi_{kn}| + |\phi_{mk}|\,|b^{2\alpha}_{kn}\psi_{kn}|\right)\\
&= c(2\alpha)\left(\|(U_{2\alpha}\widetilde{\phi})\widetilde{\psi}\|_{1,-\alpha} + \|\widetilde{\phi}(U_{2\alpha}\widetilde{\psi})\|_{1,-\alpha}\right).
\end{aligned} \tag{21}$$

Lemma 3 and a Hölder inequality now yield, for $\alpha > d$,

$$\|(U_{2\alpha}\widetilde{\phi})\widetilde{\psi}\|_{1,-\alpha} \leq C'_{-\alpha}\|(U_{2\alpha}\widetilde{\phi})\widetilde{\psi}\|_1 \leq C'_{-\alpha}\|U_{2\alpha}\widetilde{\phi}\|_2\|\psi\|_2 = C'_{-\alpha}\|\phi\|_{2,2\alpha}\|\psi\|_2.$$

Combining this with (21) gives (19). ∎

4 Decay of free solutions

The goal in this section is to prove Theorem 2. The main step in the proof is to establish appropriate time-decay estimates for the matrix elements $\left(e^{-it\boldsymbol{\Delta}}\right)_{nm,kl}$ of the free time-evolution operator with respect to the orthonormal basis $\{(|n\rangle\langle m|)\}$ for $\mathcal{H}$. This is our first objective.

Since the Weyl map W is unitary up to a constant factor by (2), the matrix elements of the heat operator $e^{-t\boldsymbol{\Delta}}$, $t > 0$, can be obtained in closed form from the well-known expression for the heat kernel in Euclidean space and the relation (4), making use of the fact that the functions $W^{-1}(|n\rangle\langle m|)$ can be computed explicitly in terms of Laguerre polynomials. The result is given in [18] and reads, in case $d = 1$,

$$\left(e^{-t\boldsymbol{\Delta}}\right)_{nm,kl} = \delta_{m+k,n+l}\sum_{v=0}^{\min\{m,l\}} C_{nm,kl,v}\frac{t^{m+l-2v}}{(1+t)^{m+k+d}},$$

where

$$C_{nm,kl,v} := \sqrt{\binom{n}{m-v}\binom{k}{l-v}\binom{m}{m-v}\binom{l}{l-v}}, \tag{22}$$

for arbitrary non-negative integers n, m, k, l. By convention the binomial coefficient $\binom{n}{m}$ vanishes unless $0 \le m \le n$. Note that the presence of the Kronecker delta factor is a consequence of rotational invariance of the heat kernel in two-dimensional space, which in the operator formulation entails that $e^{-t\boldsymbol{\Delta}}$ commutes with e^{-isa^*a} for $s \in \mathbb{R}$, where we have dropped the subscript on the annihilation and creation operators when $d = 1$.

Since $\boldsymbol{\Delta}$ is a positive, self-adjoint operator we obtain $\left(e^{-it\boldsymbol{\Delta}}\right)_{nm,kl}$ for $t \in \mathbb{R}$ by analytic continuation in t, that is

$$\left(e^{-it\boldsymbol{\Delta}}\right)_{nm,kl} = \delta_{m+k,n+l} \sum_{v=0}^{\min\{m,l\}} C_{nm,kl,v} \frac{(it)^{m+l-2v}}{(1+it)^{m+k+d}}. \tag{23}$$

We next observe that these matrix elements are expressible in terms of Jacobi polynomials.

Lemma 5. *For $d = 1$ and $l \le m, k \le n$, we have*

$$\left(e^{-it\boldsymbol{\Delta}}\right)_{nm,kl} = \delta_{m+k,n+l} \sqrt{\frac{n!l!}{m!k!}} \frac{(it)^{m+l}}{(1+it)^{m+k+1}} \left(1+t^{-2}\right)^l P_l^{n-m,m-l}\left(\frac{t^2-1}{t^2+1}\right), \tag{24}$$

where $P_l^{\alpha,\beta}(X)$, $l \in \mathbb{N}_0$, $\alpha, \beta \ge 0$, $X \in [-1, 1]$, are the classical Jacobi polynomials with standard normalization [16].

Proof. For $m + k = n + l$ and $n \ge m$ we have

$$C_{nm,kl,v} = \sqrt{\frac{n!l!}{m!k!}} \binom{m}{v} \binom{l+n-m}{l-v},$$

and consequently, for $l \le m$,

$$\left(e^{-it\boldsymbol{\Delta}}\right)_{nm,kl} = \delta_{m+k,n+l} \sqrt{\frac{n!l!}{m!k!}} \frac{(it)^{m+l}}{(1+it)^{m+k+1}} \sum_{v=0}^{l} \binom{m}{v} \binom{l+n-m}{l-v} \left(-t^{-2}\right)^v. \tag{25}$$

Recall now [16] that the classical Jacobi polynomials $P_l^{\alpha,\beta}(X)$, $l \in \mathbb{N}_0$, are orthogonal w.r.t. the weight function

$$w(X) = (1-X)^\alpha (X+1)^\beta, \qquad X \in [-1, 1], \tag{26}$$

for fixed values of $\alpha, \beta > -1$. For our purposes we may restrict attention to integer values of α and β. A convenient explicit form of $P_l^{\alpha,\beta}(X)$ with standard normalization is

$$P_l^{\alpha,\beta}(X) = \sum_{j=0}^{l} \binom{l+\alpha}{l-j} \binom{l+\beta}{j} \left(\frac{X-1}{2}\right)^j \left(\frac{X+1}{2}\right)^{l-j}.$$

With

$$X = (t^2-1)/(t^2+1) \in [-1, 1] \tag{27}$$

this gives

$$P_l^{\alpha,\beta}\left(\frac{t^2-1}{t^2+1}\right) = \left(1+t^{-2}\right)^l \sum_{j=0}^{l} \binom{l+\alpha}{l-j} \binom{l+\beta}{j} \left(-t^{-2}\right)^j$$

and the result follows by comparison with (25). ∎

The representation (24) combined with rather recent uniform estimates on the Jacobi polynomials may now be used to derive the following bound on the matrix elements in question.

Lemma 6. *For $d \geq 1$, there exists a constant C_d, independent of $n, m, k, l \in \mathbb{N}_0^d$ and $t \in \mathbb{R}$, such that*

$$\left| \left(e^{-it\mathbf{\Delta}}\right)_{nm,kl} \right| \leq C_d |t|^{-\frac{d}{2}}, \qquad |t| \geq 1. \tag{28}$$

Proof. From the definition of $\mathbf{\Delta}$ it follows that the matrix elements in question factorize into those for the one-dimensional case, so it suffices to consider $d = 1$.

Since $e^{-it\mathbf{\Delta}}$ is unitary, it follows from (23) that

$$\left(e^{-it\mathbf{\Delta}}\right)_{nm,kl} = \left(e^{-it\mathbf{\Delta}}\right)_{kl,nm}.$$

Moreover, since $W(\overline{f}) = W(f)^*$ and the heat kernel on $\mathbb{R}^2$ is symmetric in its two arguments we likewise have

$$\left(e^{-it\mathbf{\Delta}}\right)_{nm,kl} = \left(e^{-it\mathbf{\Delta}}\right)_{mn,lk}.$$

These symmetries may also be checked directly from (23) and (22).

Taking into account that $m + k = n + l$ for non-vanishing matrix elements we can therefore assume that $l \leq m, k \leq n$. Lemma 5 then yields

$$\left| \left(e^{-it\mathbf{\Delta}}\right)_{nm,kl} \right| = \delta_{m+k,n+l} \sqrt{\frac{n!l!}{m!k!}} (1+t^2)^{-1/2} \left(1 + t^{-2}\right)^{(l-n)/2} |t|^{l-k} \left| P_l^{\alpha,\beta}\left(\frac{t^2-1}{t^2+1}\right) \right|, \tag{29}$$

where we have set

$$\alpha := n - m \qquad \text{and} \qquad \beta := m - l.$$

Introducing the orthonormal Jacobi polynomials $\mathbf{P}_l^{\alpha,\beta}$ of degree $l \geq 0$ by normalizing w.r.t. the L^2-norm defined by the weight (26), one finds (see e.g. [16, p. 67])

$$\mathbf{P}_l^{\alpha,\beta}(X) = \sqrt{\frac{(2l+\alpha+\beta+1)}{2^{\alpha+\beta+1}} \frac{(l+\alpha+\beta)!l!}{(l+\alpha)!(l+\beta)!}} P_l^{\alpha,\beta}(X),$$

and (29) can be rewritten as

$$\begin{aligned} \left| \left(e^{-it\mathbf{\Delta}}\right)_{nm,kl} \right| &= \delta_{m+k,n+l} \left(l + \frac{\alpha+\beta+1}{2}\right)^{-\frac{1}{2}} \\ &\quad \times \left(1+t^2\right)^{-1/2} (1-X)^{\alpha/2} (1+X)^{\beta/2} \left|\mathbf{P}_l^{\alpha,\beta}(X)\right|, \end{aligned} \tag{30}$$

with X given by (27).

We now use the following uniform bound for the orthonormal Jacobi polynomials, proven in [5]:

$$(1-X)^{\alpha/2+1/4}(1+X)^{\beta/2+1/4}\left|\mathbf{P}_l^{\alpha,\beta}(X)\right| \leq \sqrt{\frac{2e}{\pi}} \sqrt{2 + \sqrt{\alpha^2+\beta^2}}, \tag{31}$$

valid for all $X \in]-1, 1[$, $l \geq 0$ and $\alpha, \beta \geq -1/2$.

Applying this inequality in conjunction with (30) we obtain, for $m, n, k, l \geq 0$ and $|t| \geq 1$,

$$\left| \left(e^{-it\mathbf{\Delta}}\right)_{mn,kl} \right| \leq C \delta_{m+k,n+l} \left(\frac{2 + \sqrt{\alpha^2+\beta^2}}{2l+\alpha+\beta+1}\right)^{\frac{1}{2}} |t|^{-\frac{1}{2}} \leq C' \delta_{m+k,n+l} |t|^{-\frac{1}{2}},$$

which is the announced result. ∎

In a recent article [12], Krasikov has improved the bound (31) to

$$(1-X)^{\alpha/2+1/4}(1+X)^{\beta/2+1/4}\big|\mathbf{P}_l^{\alpha,\beta}(X)\big| \le \sqrt{3}\alpha^{1/6}\left(1+\frac{\alpha}{l}\right)^{1/12},$$

valid for $X \in]-1,1[$, $l \ge 6$ and $\alpha \ge \beta \ge (1+\sqrt{2})/4$. Apart from the restricted domain of validity, which presumably can be extended to $\alpha, \beta, l \ge 0$, it is not clear whether it may lead to a stronger decay estimate than (37). In particular, we do not know whether the decay exponent $d/2$ in (28) is optimal. It is, on the other hand, easy to obtain improved uniform bounds on diagonal matrix elements as demonstrated by the following Lemma, to be used in the treatment of the diagonal case in Section 6.

Lemma 7. *For any $d \ge 1$ there exists a constant C_d', independent of $n, m \in \mathbb{N}_0^d$ and t, such that*

$$|\left(e^{-it\boldsymbol{\Delta}}\right)_{nn,mm}| \le C_d'|t|^{-d}(1+\log|t|)^d, \qquad |t| \ge 1. \tag{32}$$

Proof. Again, we may assume that $d = 1$. We then obtain from (29)

$$\left|\left(e^{-it\boldsymbol{\Delta}}\right)_{nn,mm}\right| = (1+t^2)^{-1/2}\left(1+t^{-2}\right)^{-\beta/2}\left|P_l^{0,\beta}\left(\frac{t^2-1}{t^2+1}\right)\right|, \qquad \text{with} \quad \beta = n-m,$$

and (32) will follow once we show that there exist constants C_1, C_2, independent of $\alpha, \beta, l \in \mathbb{N}_0$ and $X \in]-1,1[$, such that

$$\big|P_l^{\alpha,\beta}(X)\big| \le \left(\frac{2}{1-X}\right)^{\alpha/2}\left(\frac{2}{1+X}\right)^{\beta/2}\left(C_1 + C_2\big|\log(1-|X|)\big|\right).$$

This alternative estimate on the Jacobi polynomials relies on a very standard integral representation. We give a detailed proof for the sake of completeness. Taking into account the symmetry relation

$$P_l^{\alpha,\beta}(X) = (-1)^l\, P_l^{\beta,\alpha}(-X),$$

it suffices to consider $X \in [0,1[$. Setting $X = \cos\theta$, $\theta \in]0, \frac{\pi}{2}]$, the assertion we want to prove is equivalent to

$$\left(\sin\frac{\theta}{2}\right)^{\alpha}\left(\cos\frac{\theta}{2}\right)^{\beta}\big|P_l^{\alpha,\beta}(\cos\theta)\big| \le C_1 + C_2|\log\theta|.$$

For $X \ne \pm 1$, we can use the following integral representation, see e.g. [16, p. 70]:

$$P_l^{\alpha,\beta}(X) = \frac{2^{\alpha+\beta}}{2\pi i}\int_\Gamma \frac{dz}{z^{l+1}}\frac{R(X,z)^{-1}}{\left(1-z+R(X,z)\right)^{\alpha}\left(1+z+R(X,z)\right)^{\beta}},$$

where $R(X,z) = \sqrt{1-2Xz+z^2}$ and Γ is a positively oriented contour in the complex plane enclosing the point X. Choosing for Γ the unit circle centered at the origin, this yields

$$P_l^{\alpha,\beta}(\cos\theta) = \frac{2^{\alpha+\beta}}{2\pi}\int_{-\pi}^{\pi} d\varphi \frac{e^{-il\varphi}R(\theta,\varphi)^{-1}}{\left(1-e^{i\varphi}+R(\theta,\varphi)\right)^{\alpha}\left(1+e^{i\varphi}+R(\theta,\varphi)\right)^{\beta}},$$

where

$$R(\theta,\varphi) = \sqrt{1-2\cos\theta e^{i\varphi}+e^{2i\varphi}} = 2\sqrt{-e^{i\varphi}\sin\frac{\varphi+\theta}{2}\sin\frac{\varphi-\theta}{2}}$$

and the complex square root is defined in such a way that $\sqrt{1} = 1$. Using a trigonometric identity one finds

$$R(\theta,\varphi) = \begin{cases} 2e^{i\varphi/2}\left|\sin\frac{\varphi+\theta}{2}\sin\frac{\varphi-\theta}{2}\right|^{1/2}, & \text{if} \quad \left|\sin\frac{\varphi}{2}\right| \le \sin\frac{\theta}{2}, \\ -2ie^{i\varphi/2}\left|\sin\frac{\varphi+\theta}{2}\sin\frac{\varphi-\theta}{2}\right|^{1/2}, & \text{if} \quad \sin\frac{\varphi}{2} \ge \sin\frac{\theta}{2}, \\ 2ie^{i\varphi/2}\left|\sin\frac{\varphi+\theta}{2}\sin\frac{\varphi-\theta}{2}\right|^{1/2}, & \text{if} \quad \sin\frac{\varphi}{2} \le -\sin\frac{\theta}{2}, \end{cases}$$

and thus

$$\begin{aligned} \left|1-e^{i\varphi}+R(\theta,\varphi)\right| &= 2\left|\sin\frac{\varphi}{2}+\frac{i}{2}e^{-i\varphi/2}R(\theta,\varphi)\right| \\ &= \begin{cases} 2\left(\sin^2\frac{\varphi}{2}+\left|\sin\frac{\varphi+\theta}{2}\sin\frac{\varphi-\theta}{2}\right|\right)^{1/2} = 2\sin\frac{\theta}{2}, & \text{if} \quad \left|\sin\frac{\varphi}{2}\right| \le \sin\frac{\theta}{2}, \\ 2\left(\left|\sin\frac{\varphi}{2}\right|+\left|\sin\frac{\varphi+\theta}{2}\sin\frac{\varphi-\theta}{2}\right|\right) \ge 2\sin\frac{\theta}{2}, & \text{if} \quad \left|\sin\frac{\varphi}{2}\right| \ge \sin\frac{\theta}{2}. \end{cases} \end{aligned}$$

Similarly, one establishes

$$\left|1+e^{i\varphi}+R(\theta,\varphi)\right| \ge 2\left|\cos\frac{\theta}{2}\right|.$$

This eventually implies the following bound:

$$\begin{aligned} &\left(\sin\frac{\theta}{2}\right)^{\alpha}\left(\cos\frac{\theta}{2}\right)^{\beta}\left|P_l^{\alpha,\beta}(X)\right| \\ &\quad \le \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi\left|R(\theta,\varphi)\right|^{-1} = \frac{1}{2\pi}\int_0^{\pi} d\varphi\left|\sin\frac{\varphi+\theta}{2}\sin\frac{\varphi-\theta}{2}\right|^{-1/2}. \end{aligned}$$

Finally, it is easily seen that there exists an upper bound of the form $C_1+C_2|\log\theta|$ for the latter integral, thus completing the proof. ■

We are now ready to prove Theorem 2.

Proof of Theorem 2. Recalling the definition (7) of U_α and that b_{mn} only depends on $|n-m|$, it follows from the presence of the Kronecker-delta factor in $(e^{-it\boldsymbol{\Delta}})_{nm,kl}$ that

$$e^{-it\boldsymbol{\Delta}_\alpha}(|n\rangle\langle m|) = U_\alpha^{-1}e^{-it\boldsymbol{\Delta}}U_\alpha(|n\rangle\langle m|) = e^{-it\boldsymbol{\Delta}}(|n\rangle\langle m|)$$

is independent of α. Since $\{b_{mn}^{-\alpha}|n\rangle\langle m|\}$ is an orthonormal basis for $\mathcal{H}_\alpha$ we obtain, for $\alpha \ge 0$ and

$$\phi = \sum_{m,n}\phi_{mn}|n\rangle\langle m| \in \mathcal{H}_\alpha \subseteq \mathcal{H},$$

that

$$e^{-it\boldsymbol{\Delta}_\alpha}\phi = \sum_{n,m}\phi_{mn}e^{-it\boldsymbol{\Delta}_\alpha}(|n\rangle\langle m|) = \sum_{n,m}\phi_{mn}e^{-it\boldsymbol{\Delta}}(|n\rangle\langle m|) = e^{-it\boldsymbol{\Delta}}\phi,$$

i.e. $e^{-it\boldsymbol{\Delta}_\alpha}$ equals the restriction of $e^{-it\boldsymbol{\Delta}}$ to $\mathcal{H}_\alpha$. Applying Lemma 2 we then get, for $\beta > d$ and $\phi \in \mathcal{H}_\alpha \cap \mathcal{L}_{1,\beta}$,

$$\left\|e^{-it\boldsymbol{\Delta}_\alpha}\phi\right\|_a \le C_\beta \sup_{n,m}\left|b_{nm}^{\beta}\sum_{k,l}\left(e^{-it\boldsymbol{\Delta}}\right)_{nm,kl}\phi_{kl}\right| \le C_\beta \sup_{n,m}\sum_{k,l}\left|\left(e^{-it\boldsymbol{\Delta}}\right)_{nm,kl}b_{kl}^{\beta}\phi_{kl}\right|$$

$$\leq C_\beta \sup_{n,m,k,l} \left| \left(e^{-it\boldsymbol{\Delta}}\right)_{nm,kl} \right| \sum_{k,l} \left| b_{kl}^\beta \phi_{kl} \right| = C_\beta \sup_{n,m,k,l} \left| \left(e^{-it\boldsymbol{\Delta}}\right)_{nm,kl} \right| \|\phi\|_{1,\beta},$$

where, in the second step, we have once more made use of the Kronecker-delta factor in $(e^{-it\boldsymbol{\Delta}})_{nm,kl}$. The claim now follows from Lemma 6. ■

Remark 2. The time-decay exponent $\frac{d}{2}$ found in Lemma 6 equals half the value of the one for $2d$-dimensional Euclidean space, which is d. It is worth noting that the corresponding heat kernel actually exhibits the same decay rate as for Euclidean space. In order to see this it suffices to note that

$$C_{nm,kl,v} \leq \sqrt{\binom{n+l}{n-m+2v}\binom{m+k}{n-m+2v}} = \binom{m+k}{n-m+2v},$$

for $m+k = n+l$. Hence, for $t > 0$,

$$\begin{aligned}
\left(e^{-t\boldsymbol{\Delta}}\right)_{nm,kl} &\leq \delta_{m+k,n+l} \sum_{v=0}^{\min\{m,l\}} \binom{m+k}{n-m+2v} \frac{t^{m+l-2v}}{(1+t)^{m+k+d}} \\
&= \delta_{m+k,n+l} \frac{t^{m+k}}{(1+t)^{m+k+d}} \sum_{v=0}^{\min\{m,l\}} \binom{m+k}{n-m+2v} t^{-(n-m+2v)} \\
&\leq \frac{t^{m+k}}{(1+t)^{m+k+d}} \sum_{w=0}^{m+k} \binom{m+k}{w} t^{-w} \\
&= \frac{t^{m+k}}{(1+t)^{m+k+d}} \left(1+t^{-1}\right)^{m+k} = (1+t)^{-d}.
\end{aligned}$$

5 Existence of wave operators for small data

Proof of Theorem 3. Both statements follow from [13, Theorem 16] once we verify the following four conditions for $\alpha > 2d$ and some $\delta > 0$, where p denotes the lowest degree of monomials occurring in F:

i) There exists a constant $c_1 > 0$ such that

$$\|e^{-it\boldsymbol{\Delta}_\alpha}\phi\|_a \leq c_1 |t|^{-d/2} \|\phi\|_{1,\alpha/2}, \qquad |t| \geq 1.$$

ii) There exists a constant $c_2 > 0$ such that

$$\|\phi\|_a \leq c_2 \|\phi\|_{2,\alpha}.$$

iii) There exists a constant $c_3 > 0$ such that

$$\|\phi F(\phi^*\phi) - \psi F(\psi^*\psi)\|_{2,\alpha} \leq c_3 \big(\|\phi\|_a + \|\psi\|_a\big)^{2p-1} \|\phi - \psi\|_{2,\alpha}, \qquad \text{if } \|\phi\|_{2,\alpha}, \|\psi\|_{2,\alpha} \leq \delta.$$

iv) There exists a constant $c_4 > 0$ such that

$$\begin{aligned}
&\|\phi F(\phi^*\phi) - \psi F(\psi^*\psi)\|_{1,\alpha/2} \\
&\quad \leq c_4 \Big\{ \big(\|\phi\|_a + \|\psi\|_a\big)^{2p-2} \|\phi - \psi\|_a + \big(\|\phi\|_a + \|\psi\|_a\big)^{2p-1} \|\phi - \psi\|_{2,\alpha} \Big\}, \\
&\qquad \text{if } \|\phi\|_{2,\alpha}, \|\psi\|_{2,\alpha} \leq \delta.
\end{aligned}$$

Moreover, it is required that the constants c_3, c_4 can be chosen arbitrarily small by choosing δ small enough.

To be specific, the stated basic assumptions about F ensure that $2p-1 \geq 1$ and $\frac{d}{2}(2p-1) > 1$ for all $d \geq 1$, which implies (11) by [13, Theorem 16]. If, in addition, F has no linear term, we have $2p-1 > 1$ and (12) follows similarly.

Condition i) above is a particular case of Theorem 2. That condition ii) holds, follows from

$$\|\phi\|_a^2 \leq \|\widetilde{\phi}\|_2^2 = \|\phi\|_{2,0}^2 \leq \sum_{m,n} b_{mn}^{2\alpha} \big|\phi_{mn}\big|^2 = \|\phi\|_{2,\alpha}^2, \qquad \alpha \geq 0,$$

where we have used that the Hilbert–Schmidt norm dominates the operator norm in the first step and that $b_{mn}^{2\alpha} \geq 1$ for $\alpha \geq 0$ in the third step.

To establish iii) and iv) we make use of Lemma 4. As a consequence of the inequality $\|\phi\,\psi\|_a \leq \|\phi\|_a \|\psi\|_a$, which follows immediately from

$$(\widetilde{\phi\psi})_{mn} = \Big|\sum_k \phi_{mk}\psi_{kn}\Big| \leq \sum_k |\phi_{mk}|\,|\psi_{kn}| = (\widetilde{\phi}\widetilde{\psi})_{mn},$$

the first part of the Lemma implies

$$\|\phi_1 \cdots \phi_r\|_{2,\alpha} \leq C(\alpha, r) \sum_{i=1}^{r} \|\phi_i\|_{2,\alpha} \prod_{j \neq i} \|\phi_j\|_a, \tag{33}$$

valid for $\alpha \geq 0$ and some constant $C(\alpha, r)$ depending on α and r only.

The second statement of Lemma 4 implies, for $\alpha > 2d$,

$$\begin{aligned}\|\phi_1 \cdots \phi_r\|_{1,\alpha/2} &\leq C_{2,\alpha/2}\bigg(\|\phi_1\|_{2,\alpha}\Big\|\prod_{i=2}^{r}\phi_i\Big\|_2 + \|\phi_1\|_2\Big\|\prod_{i=2}^{r}\phi_i\Big\|_{2,\alpha}\bigg) \\ &\leq 2C_{2,\alpha/2}\|\phi_1\|_{2,\alpha}\Big\|\prod_{i=2}^{r}\phi_i\Big\|_{2,\alpha}\end{aligned}$$

where, in the second step, it has been used that $\|\phi\|_{2,\alpha}$ is an increasing function of α.

The last expression can now be estimated by making use of (33). Thus we obtain

$$\|\phi_1 \cdots \phi_r\|_{1,\alpha/2} \leq 2C(\alpha, r-1)C_{2,\alpha/2}\|\phi_1\|_{2,\alpha} \sum_{i=2}^{r} \|\phi_i\|_{2,\alpha} \prod_{j \neq 1,i} \|\phi_j\|_a. \tag{34}$$

Now, let p be an arbitrary positive integer and write

$$\phi|\phi|^{2p} - \psi|\psi|^{2p} = \sum_{i=0}^{p-1} \phi|\phi|^{2(p-1-i)}(\phi^*(\phi-\psi) + (\phi^*-\psi^*)\psi)|\psi|^{2i} + (\phi-\psi)|\psi|^{2p}.$$

Since the norms $\|\cdot\|_{2,\alpha}$, $\|\cdot\|_a$, $\|\cdot\|_{1,\alpha}$ are $*$-invariant, an application of (33) and (34) yields the inequalities

$$\|\phi|\phi|^{2p} - \psi|\psi|^{2p}\|_{2,\alpha} \leq c_3'\big(\|\phi\|_{2,\alpha} + \|\psi\|_{2,\alpha}\big)\big(\|\phi\|_a + \|\psi\|_a\big)^{2p-1}\|\phi-\psi\|_{2,\alpha},$$

and

$$\begin{aligned}\|\phi|\phi|^{2p} - \psi|\psi|^{2p}\|_{1,\alpha/2} \leq c_4'\Big\{&\big(\|\phi\|_{2,\alpha} + \|\psi\|_{2,\alpha}\big)^2\big(\|\phi\|_a + \|\psi\|_a\big)^{2p-2}\|\phi-\psi\|_a \\ &+ \big(\|\phi\|_{2,\alpha} + \|\psi\|_{2,\alpha}\big)\big(\|\phi\|_a + \|\psi\|_a\big)^{2p-1}\|\phi-\psi\|_{2,\alpha}\Big\},\end{aligned}$$

respectively, for suitable constants c_3', c_4', where the obvious inequality

$$\|\phi\|_a \leq \|\phi\|_{2,\alpha}, \qquad \alpha \geq 0,$$

has also been used. This evidently proves iii) and iv), with c_3 and c_4 of order δ, if F is a monomial of degree p. The same bounds then follow immediately for an arbitrary polynomial F without constant term, if p denotes the lowest degree of monomials occurring in F. ■

Remark 3. As pointed out in [13, Theorem 17], the assumptions of Theorem 3 also ensure the existence of a scattering operator defined for sufficiently small initial data. More precisely, if $\delta_0 > 0$ is small enough, there exists, for each $\phi_- \in \Sigma_\alpha$ with $\|\phi_-\|_{2,\alpha} \leq \delta_0$, a unique $\phi_+ \in \Sigma_\alpha$ with $\|\phi_-\|_{2,\alpha} \leq 2\delta_0$, such that the solution $\phi^-(t)$ to (9) fulfills

$$\|\phi^-(t) - e^{-it\mathbf{\Delta}_\alpha}\phi_+\|_{2,\alpha} \to 0, \qquad \text{for} \;\; t \to +\infty,$$

and the so defined mapping $S : \{\phi_- \in \Sigma_\alpha \,|\, \|\phi_-\|_{2,\alpha} \leq \delta_0\} \to \{\phi_+ \in \Sigma_\alpha \,|\, \|\phi_+\|_{2,\alpha} \leq 2\delta_0\}$, called the *scattering operator*, is injective and continuous w.r.t. the topology defined by $\|\cdot\|_{2,\alpha}$.

6 The diagonal case

In this final section we discuss briefly equation (1) when restricted to functions $\varphi(x,t)$ that correspond under the Weyl map to operators that are diagonal w.r.t. the basis $\{|n\rangle\}$. This means that the operators $\phi(t)$ commute with the number operators $a_k^* a_k$, $k = 1,\ldots,d$. Since $a_k^* a_k$ corresponds under the Weyl map to the generator of rotations in the (x_k, x_{k+d})-plane, the functions $\varphi(x,t)$ in question are invariant under such rotations, i.e. under the action of the d-fold product of $SO(2)$. Since, clearly, $\phi F(|\phi|^2)$ is diagonal and Hilbert–Schmidt if ϕ is and $\mathbf{\Delta}$ naturally restricts to a self-adjoint operator on the Hilbert subspace

$$\mathcal{H}_{\text{diag}} = \{\phi \in \mathcal{H} \,|\, \phi \text{ diagonal}\},$$

(see [4]) it follows that the equations (3) and (5) make sense as equations on $\mathcal{H}_{\text{diag}}$.

We first note that *Theorem 1 still holds with $\mathcal{H}$ replaced by $\mathcal{H}_{\text{diag}}$*. In fact, the proof of a) applies without additional changes and part b) follows likewise since the operator ϕ_0 in the proof obtained from [4] is diagonal.

Concerning Theorem 2 we have the stronger decay estimate

$$\left|\left(e^{it\mathbf{\Delta}}\right)_{nn,mm}\right| \leq C|t|^{-d}(1+\ln|t|)^d, \qquad |t| \geq 1, \tag{35}$$

from Lemma 7 above as noted previously. Note also that for $\phi \in \mathcal{H}_{\text{diag}}$ the norms $\|\phi\|_{p,\alpha}$ are independent of α and the identities

$$\|\phi\|_a = \|\widetilde{\phi}\|_{\text{op}} = \sup_n |\phi_{nn}| = \|\phi\|_{\text{op}}\,, \qquad \|\phi\|_{1,\alpha} = \sum_n |\phi_{nn}| = \|\phi\|_1$$

hold. In view of (35) the decay estimate in Theorem 2 is hence replaced by

$$\|e^{-it\mathbf{\Delta}}\phi\|_{\text{op}} \leq C|t|^{-d}(1+\log|t|)^d\|\phi\|_1, \qquad |t| \geq 1. \tag{36}$$

We therefore define the subspace of diagonal scattering states by

$$\Sigma_{\text{diag}} := \{\phi \in \mathcal{H}_{\text{diag}} \,|\, |||\phi||| < \infty\}\,,$$

where

$$|||\phi||| := \|\phi\|_2 + \sup_{|t|\geq 1} |t|^d(1+\log|t|)^{-d}\|e^{-it\mathbf{\Delta}}\phi\|_{\text{op}}.$$

Using the well known inequalities

$$\|\phi\psi\|_2 \le \frac{1}{2}\big(\|\phi\|_{\mathrm{op}}\|\psi\|_2 + \|\phi\|_2\,\|\psi\|_{\mathrm{op}}\big) \qquad \text{and} \qquad \|\phi\psi\|_1 \le \|\phi\|_2\,\|\psi\|_2$$

as a replacement for Lemma 4, we obtain the estimates

$$\begin{aligned} &\|\phi|\phi|^{2p} - \psi|\psi|^{2p}\|_2 \le C_1\big(\|\phi\|_2 + \|\psi\|_2\big)\big(\|\phi\|_{\mathrm{op}} + \|\psi\|_{\mathrm{op}}\big)^{2p-1}\|\phi-\psi\|_2, \\ &\|\phi|\phi|^{2p} - \psi|\psi|^{2p}\|_1 \le C_2\Big\{\big(\|\phi\|_2 + \|\psi\|_2\big)^2\big(\|\phi\|_{\mathrm{op}} + \|\psi\|_{\mathrm{op}}\big)^{2p-2}\|\phi-\psi\|_{\mathrm{op}} \\ &\qquad\qquad + \big(\|\phi\|_2 + \|\psi\|_2\big)\big(\|\phi\|_{\mathrm{op}} + \|\psi\|_{\mathrm{op}}\big)^{2p-1}\|\phi-\psi\|_2\Big\}, \end{aligned}$$

by arguments analogous to those in the proof of Theorem 3. It hence follows that the conclusions of Theorem 3 hold in the diagonal case with $\|\cdot\|_2$ replacing $\|\cdot\|_{2,\alpha}$ and $|||\cdot|||$ replacing $|||\cdot|||_\alpha$, provided F has no constant term and, in addition, no linear term if $d = 1$. According to the diagonal version of Theorem 2, we have in this case also global existence of solutions to the Cauchy problem on $\mathcal{H}$. It then follows by standard arguments [13, Theorem 19] that the restriction to small data at $\pm\infty$ can be dropped, i.e. the wave operators $\Omega_\pm$ can be defined on the full space Σ_{diag} if F has no linear term. More precisely we have:

Theorem 4. *Assume the polynomial F is real and contains no constant and linear terms. Then, for all $\phi_\pm \in \Sigma_{\mathrm{diag}}$, equations* (10) *and* (9) *have unique globally defined continuous solutions $\phi^\pm : \mathbb{R} \to \Sigma_{\mathrm{diag}}$ fulfilling*

$$\begin{aligned} &\|\phi^\pm(t) - e^{-it\boldsymbol{\Delta}}\phi_\pm\|_2 \to 0 \qquad \text{for} \quad t \to \pm\infty, \\ &|||e^{it\boldsymbol{\Delta}}\phi^\pm(t) - \phi_\pm||| \to 0 \qquad \text{for} \quad t \to \pm\infty. \end{aligned} \tag{37}$$

Remark 4. This result allows us to define wave operators $\Omega_\pm : \Sigma_{\mathrm{diag}} \to \Sigma_{\mathrm{diag}}$ by equation (13) that are injective and uniformly continuous on balls in Σ_{diag}. In general, $\Omega_\pm$ are not surjective: choosing F to satisfy the assumptions of Theorem 1*b*) the oscillating solution $\phi(t) = e^{i\omega t}\phi_0$ to the Cauchy problem (with $t_0 = 0$) fulfills $\|\phi(t)\|_1 = \|\phi_0\|_1 < \infty$ by [4, Lemma 2]. Hence, $\phi_0 \in \Sigma_{\mathrm{diag}}$ by (36). On the other hand, since the unique solution $\phi(t)$ to the Cauchy problem with initial value ϕ_0 at $t = 0$ has constant operator norm, it cannot fulfill (37) for any $\phi_\pm \in \Sigma_{\mathrm{diag}}$. Even in this case, an appropriate characterization of the images of $\Omega_\pm$ remains an open question.

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